Acta Math. Univ. Comenianae
Vol. LXXXIX, 2 (2020), pp. 295–298

ON THE CONTINUOUS DUAL OF THE SEQUENCE SPACE bv

M. EL AZHARI

Abstract. Imaninezhad and Miri introduced the sequence space d_∞ in order to characterize the continuous dual of the sequence space bv. We show by a counterexample that this claimed characterization is false.

1. Preliminaries

Let ω denote the vector space of all complex sequences, where addition and scalar multiplication are defined pointwise. For each $k \geq 1$, let $e^{(k)} = (e_n^{(k)})_{n\geq 1}$ be the complex sequence defined by $e_k^{(k)} = 1$ and $e_n^{(k)} = 0$ for $n \neq k$. A sequence space is a vector subspace of ω including the set $\{e^{(k)} : k \geq 1\}$. By l_∞, c, and l_q $(1 \leq q < \infty)$, we denote the sequence spaces of all bounded, convergent and absolutely q-summable sequences, respectively. Also the inclusions $l_q \subset c \subset l_\infty$ are strict. c and l_∞ are Banach spaces with the supremum norm $\|x\|_\infty = \sup\{|x_k| : k \geq 1\}$. l_q is a Banach space with the norm $\|x\|_q = (\sum_{k=1}^{\infty}|x_k|^q)^{\frac{1}{q}}$.

Let bv_p $(1 \leq p < \infty)$ denote the sequence space of all sequences of bounded variation defined, in [**2**], by $bv_p = \{x = (x_k)_{k\geq 1} \in \omega : \sum_{k=1}^{\infty}|x_k - x_{k-1}|^p < \infty\}$ with $x_0 = 0$. bv_p is a Banach space with the norm $\|x\|_{bv_p} = (\sum_{k=1}^{\infty}|x_k - x_{k-1}|^p)^{\frac{1}{p}}$. We write bv instead of bv_1. If we define the sequence $b^{(k)} = (b_n^{(k)})_{n\geq 1}$ of elements of the space bv_p for every $k \geq 1$ by $b_n^{(k)} = 0$ if $n < k$ and $b_n^{(k)} = 1$ if $n \geq k$, then the sequence $(b^{(k)})_{k\geq 1}$ is a Schauder basis [**2**] for bv_p and every $x \in bv_p$ has a unique representation of the form $x = \sum_{k=1}^{\infty}(x_k - x_{k-1})b^{(k)}$ with $x_0 = 0$.

Let Δ_ω denote the difference operator on ω defined by $\Delta_\omega x = (x_k - x_{k-1})_{k\geq 1}$ with $x_0 = 0$ for every $x = (x_k)_{k\geq 1} \in \omega$. For each $1 \leq p < \infty$, the map $\Delta\colon bv_p \to l_p$, $\Delta x = \Delta_\omega x$, is an isometric linear isomorphism [**2**]. If $(E, \|.\|)$ is a normed sequence space, then E^* denotes the continuous dual of E with the norm defined by $\|f\| = \sup\{|f(x)| : \|x\| = 1\}$ for all $f \in E^*$. It is well-known that $l_1^* \cong l_\infty$ and $l_p^* \cong l_q$ $(1 < p, q < \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Received May 8, 2019; revised July 5, 2019.
2010 *Mathematics Subject Classification.* Primary 46B10, 46B45.
Key words and phrases. Continuous dual; sequence space.

2. The sequence space d_∞

Proposition 2.1. *A linear functional f on bv is continuous if and only if there exists $a = (a_k)_{k\geq 1} \in l_\infty$ such that $f(x) = \sum_{k=1}^{\infty} a_k(\Delta x)_k$ for all $x \in bv$. Furthermore, $f(b^{(i)}) = a_i$ and $f(e^{(i)}) = a_i - a_{i+1}$ for every $i \geq 1$.*

Proof. The map $\Delta\colon bv \to l_1, \Delta x = (x_k - x_{k-1})_{k\geq 1}$ with $x_0 = 0$, is an isometric linear isomorphism [**2**]. Let f be a linear functional on bv. If f is continuous, then $f \circ \Delta^{-1}$ is a continuous linear functional on l_1, so there exists $a = (a_k)_{k\geq 1} \in l_\infty$ such that $f \circ \Delta^{-1}(y) = \sum_{k=1}^{\infty} a_k y_k$ for all $y \in l_1$. Hence $f(x) = (f \circ \Delta^{-1})(\Delta x) = \sum_{k=1}^{\infty} a_k(\Delta x)_k$ for all $x \in bv$.

Conversely, if $f(x) = \sum_{k=1}^{\infty} a_k(\Delta x)_k$ for all $x \in bv$ and some $(a_k)_{k\geq 1} \in l_\infty$, then $|f(x)| \leq \sum_{k=1}^{\infty} |a_k||(\Delta x)_k| \leq \|a\|_\infty \sum_{k=1}^{\infty} |(\Delta x)_k| = \|a\|_\infty \|x\|_{bv}$. Therefore f is a continuous linear functional on bv. Let $i \geq 1, f(b^{(i)}) = \sum_{k=1}^{\infty} a_k(\Delta b^{(i)})_k = a_i$ and $f(e^{(i)}) = \sum_{k=1}^{\infty} a_k(\Delta e^{(i)})_k = a_i - a_{i+1}$. $\square$

Proposition 2.2. *bv^* is isometrically isomorphic to l_∞.*

Proof. Define $\varphi\colon bv^* \to l_\infty$, $\varphi(f) = (f(b^{(k)}))_{k\geq 1}$, φ is a surjective linear map by Proposition 2.1, and φ is injective since $(b^{(k)})_{k\geq 1}$ is a Schauder basis for bv. Let $f \in bv^*$ and $x \in bv$, $|f(x)| = |f(\sum_{k=1}^{\infty} (\Delta x)_k b^{(k)})| = |\sum_{k=1}^{\infty} (\Delta x)_k f(b^{(k)})| \leq \sum_{k=1}^{\infty} |(\Delta x)_k||f(b^{(k)})| \leq \sup_{k\geq 1}|f(b^{(k)})| \sum_{k=1}^{\infty} |(\Delta x)_k| = \|\varphi(f)\|_\infty \|x\|_{bv}$, then $\|f\| \leq \|\varphi(f)\|_\infty$.

On the other hand, $|f(b^{(k)})| \leq \|f\|\|b^{(k)}\|_{bv} = \|f\|$ since $\|b^{(k)}\|_{bv} = 1$ for all $k \geq 1$, then $\|\varphi(f)\|_\infty = \sup_{k\geq 1}|f(b^{(k)})| \leq \|f\|$. $\square$

Imaninezhad and Miri [**3**] introduced the sequence space $d_\infty = \{x = (x_k)_{k\geq 1} \in \omega : \sup_{k\geq 1}|\sum_{i=k}^{\infty} x_i| < \infty\}$, and claimed that $T\colon bv^* \to d_\infty$, $T(f) = (f(e^{(k)})_{k\geq 1}$, is an isometric linear isomorphism [**3**, Theorem 3.3] (see also [**4**, Theorem 4.1]). Here we show that the set $T(bv^*)$ is not included in d_∞.

Counterexample. Let $a = (a_k)_{k\geq 1} \in l_\infty \setminus c$, and consider the map $f(x) = \sum_{k=1}^{\infty} a_k(\Delta x)_k$ for all $x \in bv$. By Proposition 2.1, $f \in bv^*$ and $f(e^{(i)}) = a_i - a_{i+1}$ for every $i \geq 1$. Let $k \geq 1$ and $n \geq k$, $\sum_{i=k}^{n} f(e^{(i)}) = \sum_{i=k}^{n}(a_i - a_{i+1}) = a_k - a_{n+1}$. If $(f(e^{(k)}))_{k\geq 1} \in d_\infty$, then $\lim_{n\to\infty} \sum_{i=k}^{n} f(e^{(i)}) = \sum_{i=k}^{\infty} f(e^{(i)}) \in \mathbb{C}$ and consequently $\lim_{n\to\infty} a_{n+1} = a_k - \sum_{i=k}^{\infty} f(e^{(i)})$, which contradicts the fact that the sequence $(a_k)_{k\geq 1}$ is not convergent.

Remark. The above counterexample is also a counterexample to the equality $\sum_{i=k}^{\infty} f(e^{(i)}) = f(b^{(k)})$ for all $k \geq 1$ and $f \in bv^*$, which is used in the proofs of [**1**, Theorem 2.3] and [**3**, Theorem 3.3].

3. The sequence space d_q $(1 < q < \infty)$

Proposition 3.1. *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. A linear functional f on bv_p is continuous if and only if there exists $a = (a_k)_{k\geq 1} \in l_q$ such that $f(x) =$*

$\sum_{k=1}^{\infty} a_k(\Delta x)_k$ *for all* $x \in bv_p$. *Furthermore,* $f(b^{(i)}) = a_i$ *and* $f(e^{(i)}) = a_i - a_{i+1}$ *for every* $i \geq 1$.

Proof. The map $\Delta\colon bv_p \to l_p, \Delta x = (x_k - x_{k-1})_{k\geq 1}$ with $x_0 = 0$, is an isometric linear isomorphism [**2**]. Let f be a linear functional on bv_p. If f is continuous, then $f \circ \Delta^{-1}$ is a continuous linear functional on l_p, so there exists $a = (a_k)_{k\geq 1} \in l_q$ such that $f \circ \Delta^{-1}(y) = \sum_{k=1}^{\infty} a_k y_k$ for all $y \in l_p$. Hence $f(x) = (f \circ \Delta^{-1})(\Delta x) = \sum_{k=1}^{\infty} a_k(\Delta x)_k$ for all $x \in bv_p$.

Conversely, if $f(x) = \sum_{k=1}^{\infty} a_k(\Delta x)_k$ for all $x \in bv_p$ and some $a = (a_k)_{k\geq 1} \in l_q$, then $|f(x)| \leq \sum_{k=1}^{\infty}|a_k||(\Delta x)_k| \leq (\sum_{k=1}^{\infty}|a_k|^q)^{\frac{1}{q}}(\sum_{k=1}^{\infty}|(\Delta x)_k|^p)^{\frac{1}{p}} = \|a\|_q\|x\|_{bv_p}$ by Hölder inequality. Therefore f is a continuous linear functional on bv_p. $\square$

Proposition 3.2. *Let* $1 < p, q < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* bv_p^* *is isometrically isomorphic to* l_q.

Proof. Define $\psi\colon bv_p^* \to l_q$, $\psi(f) = (f(b^{(k)}))_{k\geq 1}$, ψ is a surjective linear map by Proposition 3.1, and ψ is injective since $(b^{(k)})_{k\geq 1}$ is a Schauder basis for bv_p. Let $f \in bv_p^*$ and $x \in bv_p$, $|f(x)| = |f(\sum_{k=1}^{\infty}(\Delta x)_k b^{(k)})| = |\sum_{k=1}^{\infty}(\Delta x)_k f(b^{(k)})| \leq \sum_{k=1}^{\infty}|(\Delta x)_k||f(b^{(k)})| \leq (\sum_{k=1}^{\infty}|(\Delta x)_k|^p)^{\frac{1}{p}}(\sum_{k=1}^{\infty}|f(b^{(k)})|^q)^{\frac{1}{q}} = \|\psi(f)\|_q\|x\|_{bv_p}$ by Hölder inequality, then $\|f\| \leq \|\psi(f)\|_q$.

On the other hand, let $f \in bv_p^*$ and define the sequence $(x^{(n)})_{n\geq 1}$ of the space bv_p by $(\Delta x^{(n)})_k = |f(b^{(k)})|^q f(b^{(k)})^{-1}$ if $1 \leq k \leq n$ and $f(b^{(k)}) \neq 0$, $(\Delta x^{(n)})_k = 0$ otherwise. Let $n \geq 1$, $f(x^{(n)}) = f(\sum_{k=1}^{\infty}(\Delta x^{(n)})_k b^{(k)}) = \sum_{k=1}^{\infty}(\Delta x^{(n)})_k f(b^{(k)}) = \sum_{k=1}^{n}|f(b^{(k)})|^q$.

Then $\sum_{k=1}^{n}|f(b^{(k)})|^q = f(x^{(n)}) = |f(x^{(n)})| \leq \|f\|\|x^{(n)}\|_{bv_p} = \|f\|(\sum_{k=1}^{n}|f(b^{(k)})|^{(q-1)p})^{\frac{1}{p}} = \|f\|(\sum_{k=1}^{n}|f(b^{(k)})|^q)^{\frac{1}{p}}$ since $(q-1)p = q$. Therefore $(\sum_{k=1}^{n}|f(b^{(k)})|^q)^{\frac{1}{q}} = (\sum_{k=1}^{n}|f(b^{(k)})|^q)^{1-\frac{1}{p}} \leq \|f\|$ for all $n \geq 1$. Letting $n \to \infty$, we get $\|\psi(f)\|_q = (\sum_{k=1}^{\infty}|f(b^{(k)})|^q)^{\frac{1}{q}} \leq \|f\|$. $\square$

Akhmedov and Başar [**1**] introduced the sequence space $d_q = \{x = (x_k)_{k\geq 1} \in \omega : \sum_{k=1}^{\infty}|\sum_{i=k}^{\infty} x_i|^q < \infty\}$ $(1 < q < \infty)$, with the norm $\|x\|_{d_q} = (\sum_{k=1}^{\infty}|\sum_{i=k}^{\infty} x_i|^q)^{\frac{1}{q}}$, and proved that the sequence space d_q is isometrically isomorphic to bv_p^* $(1 < p < \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ (see [1, Theorem 2.3] or [3, Theorem 3.6]). Here we deduce this result as a consequence of Proposition 3.2.

Corollary 3.3. *Let* $1 < p, q < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* bv_p^* *is isometrically isomorphic to* d_q.

Proof. By Proposition 3.2 and the fact that the map $D\colon d_q \to l_q$, $D((x_k)_{k\geq 1}) = (\sum_{i=k}^{\infty} x_i)_{k\geq 1}$, is an isometric linear isomorphism. $\square$

Remark. By using the same above argument, we cannot show that bv^* is isometrically isomorphic to d_∞ since the linear map $\delta\colon d_\infty \to l_\infty$, $\delta((x_k)_{k\geq 1}) = (\sum_{i=k}^{\infty} x_i)_{k\geq 1}$, which is injective and norm-preserving, is not surjective (we easily show that the element $(1, 1, 1, \dots) \in l_\infty$ does not belong to the image of δ).

References

1. Akhmedov A. M. and Başar F., *The fine spectra of the difference operator* Δ *over the sequence space* bv_p $(1 \le p < \infty)$, Acta Math. Sin. Eng. Ser. **23**(10) (2007), 1757–1768.
2. Başar F. and Altay B., *On the space of sequences of p-bounded variation and related matrix mappings*, Ukrainian Math. J. **55**(1) (2003), 136–147.
3. Imaninezhad M. and Miri M., *The dual space of the sequence space* bv_p $(1 \le p < \infty)$, Acta Math. Univ. Comenianae **79**(1) (2010), 143–149.
4. Imaninezhad M. and Miri M., *The continuous dual of the sequence space* $l_p(\Delta^n)$, Acta Math. Univ. Comenianae **79**(2) (2010), 273–280.

M. El Azhari, Ecole Normale Supérieure, Avenue Oued Akreuch, Takaddoum, BP 5118, Rabat, Morocco,

e-mail: `mohammed.elazhari@yahoo.fr`