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ON RELAXED ŠOLTÉS'S PROBLEM

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Abstract. The *Wiener index* is a graph parameter originating from chemical graph theory. It is defined as the sum of the lengths of the shortest paths between all pairs of vertices in given graph. In 1991, Šoltés posed the following problem regarding Wiener index. Find all graphs such that its Wiener index is preserved upon removal of any vertex. The problem is far from being solved and to this day, only one such graph is known – the cycle graph on 11 vertices.

In this paper we solve a relaxed version of the problem, proposed by Knor, Majstorović and Škrekovski. The problem is to find for a given k (infinitely many) graphs such that they have exactly k vertices such that if we remove any one of them, the Wiener index stays the same. We call such vertices *good* vertices and we show that there are infinitely many cactus graphs with exactly k cycles of length at least 7 that contain exactly $2k$ good vertices and infinitely many cactus graphs with exactly k cycles of length $c \in \{5, 6\}$ that contain exactly k good vertices. On the other hand, we prove that G has no good vertex if the length of the longest cycle in G is at most 4.

1. Introduction

The *Wiener index* (also *Wiener number*) is a topological index of a connected graph, defined as the sum of the lengths of the shortest paths between all unordered pairs of vertices in the graph. In other words, for a connected graph $G = (V, E)$, the Wiener index $W(G)$ is defined as

$$W(G) := \sum_{\{u,v\} \subseteq V} \operatorname{dist}_G(u, v).$$

The index was originally introduced in 1947 by Wiener [**9**] for the purpose of determining the approximation formula for the boiling point of paraffin. Since then, Wiener index has become one of the most frequently used topological indices in chemistry, since molecules are usually modeled by undirected graphs. The definition of Wiener index in terms of distances between vertices of a graph was first given by Hosoya [**3**]. Since then, a lot of mathematicians have studied this quantity very extensively. Apart from pure mathematics, Wiener index has many applications in chemistry, cryptography, theory of communication, topological networks etc. The quantity is used in sociometry and the theory of social networks, since

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it provides a robust measure of the network topology [**2**]. We refer the interested reader to the numerous surveys, e.g. [**1, 4, 7**].

In 1991, Šoltés posed the following problem in [**8**].

Problem 1 (Šoltés's problem)**.** Find all graphs G such that the equality $W(G) = W(G - v)$ holds for all their vertices $v \in V(G)$.

Graphs which satisfy condition $W(G) = W(G - v)$ for all $v \in V(G)$ are called *Šoltés's graphs*. Šoltés found just one such graph – C_{11}. To this day, this is the only known graph and it is not known if there is any other. Thus the Šoltés's problem is still open. Knor, Majstorović and Škrekovski defined and studied a relaxed version of Šoltés's problem.

Problem 2. Find all graphs G for which the equality $W(G) = W(G-v)$ holds for at least one vertex $v \in V(G)$.

We call $v \in V(G)$ *good vertex* if $W(G) = W(G-v)$ holds. In this terminology, a graph is a Šoltés's graph if all its vertices are good. It was shown in [**6**] that there exist infinitely many unicyclic graphs with one and two good vertices of degree 2. In [**5**], the same authors found for given $k \geq 3$ infinitely many graphs which have a good vertex of degree k and infinitely many graphs with a good vertex of degree $n - 2$ and $n - 1$. Furthermore, they proved that dense graphs cannot be Šoltés's graphs. They also posed the following problem in [**5**].

Problem 3. For given k, find infinitely many graphs G for which the equality

$$W(G) = W(G - v_1) = W(G - v_2) = \cdots = W(G - v_k)$$

holds for distinct vertices $v_1, \ldots, v_k \in V(G)$.

In this paper we solve this problem by finding such infinite class of graphs within the class of *cacti*. We recall that *cactus* is a graph where every edge belongs to at most one cycle. Let us now summarize our main results.

- We found infinitely many cactus graphs with exactly k cycles of length at least 7 that contain exactly $2k$ good vertices and infinitely many cactus graphs with exactly k cycles of length $c \in \{5, 6\}$ that contain exactly k good vertices (Theorem 2).
- We prove that G has no good vertex if the length of the longest cycle in G is at most 4 (Theorem 3).

2. Preliminaries

All graphs in this paper are simple and undirected. As our results refine and extend those of [**6**] and [**5**], most of the time we follow the notation introduced there.

Let G be a connected graph and let v be a vertex in $V(G)$. By $d_G(v)$ we denote the degree of v in G. A *pendant vertex* is a vertex of degree one and a *pendant edge* is the only edge incident to a pendant vertex. Note that Wiener index can also be written as $W(G) = \frac{1}{2}\sum_{v \in V(G)} t_G(v)$, where $t_G(v)$, the *transmission* of v in G, is the sum of distances between v and all the other vertices of G.

The complete graph K_n has the smallest Wiener index among all graphs on n vertices since the distance between any two distinct vertices is at least one in any graph. It is well known that for any connected graph on n vertices, the maximum Wiener index is obtained for the path P_n. Thus, for every graph G on n vertices we have

$$\binom{n}{2} = W(K_n) \le W(G) \le W(P_n) = \binom{n+1}{3}.$$

It is easy to see that for the Wiener index of the cycle of length n holds

$$W(C_n) = \begin{cases} \frac{n^3}{8} & \text{if } n \text{ is even,} \\ \frac{n^3-n}{8} & \text{if } n \text{ is odd.} \end{cases}$$

The proof of the following proposition is also straightforward.

Proposition 1. *Let G be a connected graph. Take a new vertex z and connect it by a pendant edge to a vertex $u \in V(G)$. Denote the resulting graph by G^+. Then $W(G^+) = W(G) + t_G(u) + n(G)$.*

Recall that v is a *good* vertex in G if $W(G) = W(G - v)$. Let v_1 be a fixed vertex in G. For a vertex $x \in V(G)$, $x \neq v_1$ we denote (similarly as in [**6**])

$$\delta_G(x) = t_G(x) - t_{G-v_1}(x) \quad \text{and} \quad \Delta(G) = W(G) - W(G - v_1).$$

Observe that $\Delta(G) = 0$ means that v_1 is a good vertex in G and $\delta_G(x)$ gives us the contribution of the vertex x to $\Delta(G)$.

3. Main results

3.1. Infinite families

We first need to state a few simple lemmata. We need them for the proof of Theorem 1.

Lemma 1. *Let G be a connected graph with a fixed vertex v_1. Take a new vertex z and connect it by a pendant edge to a vertex $u \in V(G)$, $u \neq v_1$. Denote the resulting graph by G^+. Then $\delta_{G^+}(z) = \delta_{G^+}(u) + 1 = \delta_G(u) + 1$.*

Lemma 2. *Let G be a connected graph with a fixed vertex w. Take a cycle C_c of length $c \ge 7$ and connect it to G by identifying one vertex on the cycle with w and denote the resulting graph G^*. Let v_1 be a neighbor of w on the cycle C_c. Then $\delta_{G^*}(w) = t_{G^*}(w) - t_{G^*-v_1}(w) \le -2$.*

Lemma 3. *Let G be a connected graph with a fixed vertex w. Take a cycle C_c of length $c \in \{5, 6\}$ and connect it to G by identifying one vertex on the cycle with w. Let v_2 be a vertex in distance 2 from w on the cycle C_c and let v_1 be the only common neighbor of w and v_2 on the cycle C_c. Add a path of length 2 to C_c by identifying one of its endpoints with v_2 and denote the resulting graph G^*. Then $\delta_{G^*}(w) \le -2$.*

The following theorem is the main step towards proving the main result of this paper and so we include a full proof of it.

Theorem 1. *Let c, k be natural numbers, $c \in \{5, 6\}$. There exist infinitely many cactus graphs with exactly k cycles of length c that contain at least k good vertices. If $c \geq 7$ then there exist infinitely many cactus graphs with exactly k cycles of length c that contain at least $2k$ good vertices.*

Proof. Our construction uses similar techniques as in [**6**]. We proceed in four steps by constructing graphs G_1, G_2, G_3 and G_4. The choice of G_1 is different for $c \in \{5, 6\}$ and for $c \geq 7$. Therefore, we distinguish two cases.

Case 1 ($c \in \{5, 6\}$). Let H be a cycle C_c with a path of length 2 attached to it by identifying one of its endpoints with a vertex on the cycle. Take k copies of H and denote them $H_1, \ldots, H_k$. Fix a vertex $v_0^i \in V(H_i)$ in distance two from the only vertex of degree 3 in H_i. Join $H_1, \ldots, H_k$ together by identifying all v_0^i and denote this new vertex by w. Denote the resulting graph G_1 and denote by v_1 the only common neighbor of w and the vertex of degree 3 on C_c.

Case 2 ($c \geq 7$). Take k copies of a cycle C_c, fix a vertex in each copy and identify all fixed vertices to one vertex w. Denote the resulting graph G_1 and denote by v_1 any neighbor of w in G_1.

Note that in both cases $\delta_{G_1}(w) \leq -2$, by Lemma 2 and Lemma 3.

Set $d := -\delta_{G_1}(w)$ and let $P^d := u_d, u_{d-1}, \ldots, u_1, u_0$ be a path of length d. Note that $d \geq 2$. Attach P^d to w by identifying u_d with w and denote the resulting graph G_2. The crucial observation follows immediately by iterative use of Lemma 1, namely $\delta_{G_2}(u_i) = -i$. In other words, the value of $\delta_{G_2}(u_i)$ increases along the path P^d from $\delta_{G_2}(w) = -d$ to $\delta_{G_2}(u_0) = 0$.

- If $\Delta(G_2) = 0$ we set $G_3 := G_2$.
- If $\Delta(G_2) < 0$ we connect exactly $-\Delta(G_2)$ new pendant vertices to u_0 in G_2 and denote the resulting graph G_3. As $\delta_{G_2}(u_0) = 0$, by Lemma 1 the contribution $\delta_{G_3}(x)$ of any pendant vertex $x \in V(G_3) \setminus V(G_2)$ to $\Delta(G_3)$ is $\delta_{G_3}(x) = 1$ and thus $\Delta(G_3) = 0$.
- If $\Delta(G_2) > 0$ we connect exactly $\Delta(G_2)$ new pendant vertices to u_2 in G_2 and denote the resulting graph G_3. As $\delta_{G_2}(u_2) = -2$, by Lemma 1 the contribution $\delta_{G_3}(x)$ of any pendant vertex $x \in V(G_3) \setminus V(G_2)$ to $\Delta(G_3)$ is $\delta_{G_3}(x) = -1$ and thus again $\Delta(G_3) = 0$.

Finally, for arbitrary $p \geq 0$ we add to G_3 exactly p new pendant vertices, connect them all to u_1 and denote the resulting graph G_4. As $\delta_{G_3}(u_1) = -1$, by Lemma 1 we get that for every $x \in V(G_4) \setminus V(G_3)$ the contribution $\delta_{G_4}(x) = 0$ and thus $\Delta(G_4) = \Delta(G_3) = 0$. In other words, v_1 is a good vertex in G_4 for any choice of p.

It remains to show that v_1 is not the only good vertex in G. This follows immediately from the symmetry of the starting graph G_1. It is obvious that for $c \in \{5, 6\}$, there are other $k - 1$ good vertices other than v_1 (one in each copy of C_c) since we can find one vertex in each cycle such that its removal yields a graph isomorphic to $G_4 - v_1$. If $c \geq 7$ we can argue similarly that in G_4 there are $2k$ good vertices (two in each copy of C_c). $\square$

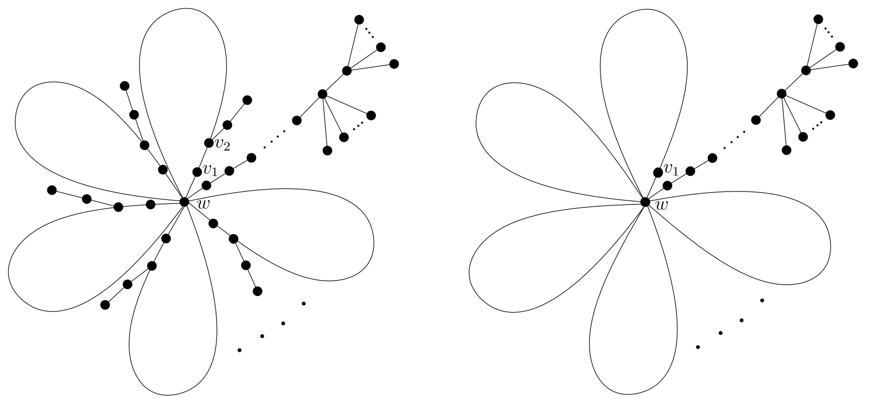

Figure 1. An illustration of construction from Theorem 1. Graphs G_4 for $c \in \{5, 6\}$ are on the left and for $c \geq 7$ on the right.

So far, we proved that for every natural k there are infinitely many graphs with at least k (or $2k$) good vertices. Now we can state the main result of the paper which says that the graphs constructed in Theorem 1 contain no other good vertices.

Theorem 2. *Let k be a natural number. Then the following holds.*

1. *For every $c \in \{5, 6\}$ there are infinitely many cactus graphs with exactly k cycles of length c that contain exactly k good vertices.*
2. *For every $c \geq 7$ there are infinitely many cactus graphs with exactly k cycles of length c that contain exactly $2k$ good vertices.*

Let us remark that if we define G_1 for $c \geq 7$ as it is done for $c \in \{5, 6\}$ (that is we add a path of length 2 to the vertex v_2), we would obtain graphs with exactly k good vertices also for $c \geq 7$ and for arbitrary k.

Furthermore, for $k = 1$ we obtain precisely the graphs constructed in [**6**]. It follows from our results that their unicyclic graphs have exactly one good vertex if the length c of the unique cycle is 5 or 6 and exactly two good vertices in the case when $c \geq 7$. We note that this fact was not proved in [**6**] and only the existence of at least one good vertex was shown there.

3.2. Negative results

The following theorem explains why we cannot hope for a similar result when the cycle length c equals 3 or 4.

Theorem 3. *Let G be a connected graph which is not a tree. If the length of the longest cycle in G is at most 4, then G has no good vertex.*

Proof. Suppose for contradiction that G has a vertex v such that $W(G) = W(G - v)$. It is obvious that v has to lie on a cycle, otherwise $G - v$ would be a disconnected graph. Note that by deleting v from G the distance between each pair of vertices in $G - v$ remains the same as in G. It follows that $W(G - v) = W(G) - t_G(v)$ and hence $W(G - v) < W(G)$, a contradiction. $\square$

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