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ε-COLORINGS OF STRIPS

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Abstract. A special case of the Hadwiger-Nelson problem is to color a strip instead of the whole plane. The aim is to maximize the width of the strip such that it still permits a coloring with c colors. We present a coloring that improves the recently best known value for 4 colors. This is discovered by considering colorings that satisfy slightly stronger distance conditions. Moreover, we can show under a sensible assumption that this value is best possible for the stronger version of the distance conditions.

1. Introduction

How many colors are needed to color the plane such that any two points of distance 1 are colored differently? That is the well-known Hadwiger-Nelson problem. It is easy to state but quite hard to solve. Since its inception in 1950 it is known that seven colors suffice, and since the breakthrough result of de Grey [**5**] in 2018 it is known that at least 5 colors are necessary. After this result the Polymath16 project [**7**] was launched in 2018 in order to narrow down the remaining gap. Despite all the activity, the problem still remains unsolved. Because of that, several special cases were proposed to obtain partial results and make further progress.

One proposed special case, for instance, consists in fixing the number of colors c and to try to color a strip of width w and infinite length [**2, 1**]. The new task is to find the largest width $\omega(c)$ such that this is still possible. Determining the smallest number of colors such that $\omega(c) = \infty$ is equivalent to the Hadwiger-Nelson problem. For $c \leq 3$ and $c \geq 7$ the values of $\omega(c)$ are known. For $c \in \{4, 5, 6\}$ lower bounds for $\omega(c)$ have been obtained by providing valid colorings.

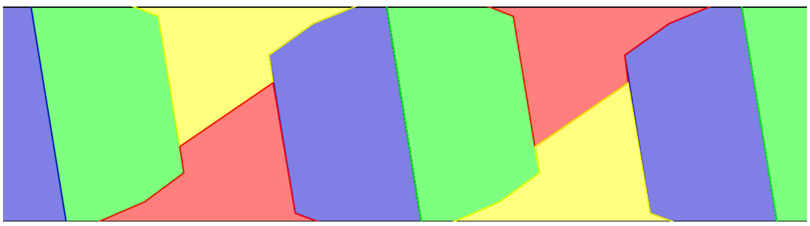

Figure 1. Coloring of a strip of width α with 4 colors.

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For $c = 4$, the smallest unknown number of colors, Bauslaugh [**2**] showed that $\omega(4) \geq \frac{\sqrt{8}}{3} \approx 0.943$ and during the Polymath16 project [**7**] this bound was improved to $\omega(4) \geq \sqrt{\frac{32}{35}} \approx 0.956$. In Figure 1 we present a coloring that implies $\omega(4) \geq \alpha \approx 0.959$, where the exact value of α is the solution of a quadratic program.

In view of the Hadwiger-Nelson problem, upper bounds on $\omega(c)$ would be more interesting than lower bounds. Unfortunately, it seems that finding upper bounds for $\omega(c)$ is as hard as the original problem.

A common approach to the general Hadwiger-Nelson problem is to require the colorings to satisfy additional properties that are naturally found in colorings of the plane. For instance, the colorings may be required to be periodic, or tile based. Exoo [**6**] and Currie and Eggleton [**3**] considered ε-colorings. These are colorings where the condition, that two point must receive different colors if their distance is exactly 1, is strengthened to they must receiving different colors if their distance is 1 up to a tolerance of ε. The introduction of ε-colorings seems to be plausible since all of the known exact colorings can easily be modified to be ε-colorings. Furthermore, it turns out that ε-colorings are much more easy to handle. An interesting result is that an ε-coloring of the plane needs at least 6 colors [**3**]. As a consequence, either every exact coloring of the plane needs at least 6 colors or there is a coloring with 5 colors that exploits heavily that the distance condition concerns distances exactly equal to 1.

Our approach is to color strips of the plane with ε-colorings. In analogy to $\omega(c)$ for exact colorings, we define $\omega_\varepsilon(c)$ for ε-colorings and write $\omega'(c) = \lim_{\varepsilon\to 0} \omega_\varepsilon(c)$. Since all of the known lower bounds for $\omega(c)$ are obtained by colorings that are limits of ε-colorings, these bounds can be reused for $\omega'(c)$. We show upper bounds for $\omega'(c)$, namely $\omega'(4) \leq 1$ and $\omega'(5) \leq \sqrt{3}$. Under a plausible assumption we can show that, $\omega'(4) \leq \alpha \approx 0.959$. Together with our coloring with 4 colors this implies our main result.

Theorem 1. $\omega'(4) = \alpha \approx 0.959$ *if* Assumption 10 *holds.*

2. Properties of ε-colorings

In order to make the colorings easier to handle, we introduce ε-colorings by making the distance condition slightly more restrictive.

Definition 2. An *ε-coloring* of a strip (or the plane) is a coloring of the points such that every two points x, y with $1 - \varepsilon \leq \mathrm{dist}(x, y) \leq 1 + \varepsilon$ receive different colors. Furthermore, we define

$\omega_\varepsilon(c) = \sup\{w | \exists\ \varepsilon\text{-coloring with } c \text{ colors of a strip of width } w.\}$ and $\omega'(c) = \lim_{\varepsilon\to 0} \omega_\varepsilon(c)$.

From the definition it is immediately clear that for $\varepsilon_1 \geq \varepsilon_2$, we get

$$\omega_{\varepsilon_1}(c) \leq \omega_{\varepsilon_2}(c) \leq \omega'(c) \leq \omega(c).$$

Compared to exact colorings, ε-colorings provide some useful features which will be examined in the following lemmas. The first lemma will show that we can restrict ε-colorings to consist of tiles which will allow us to talk of color regions.

Lemma 3. *Restricting ε-colorings to consist of unicolored regular hexagons of side length $\frac{\varepsilon}{4}$ does not change the value of $\omega'(c)$.*

The next two lemmas will provide tools that will be useful in upcoming proofs.

Lemma 4. *Let C be an ε-coloring and let s, t be two points with $\operatorname{dist}(s,t) \geq 1$. If Γ is a continuous $s-t$ curve using only two colors, then s and t have distinct colors.*

Lemma 5. *Let C be an ε-coloring and let Γ_1 be a continuous $u-v$ curve and Γ_2 be a continuous $x-y$ curve such that there are only two colors used for $\Gamma_1 \cup \Gamma_2$. If $\operatorname{dist}(u,x) \leq 1$, $\operatorname{dist}(u,y) \leq 1$, $\operatorname{dist}(v,x) \geq 1$ and $\operatorname{dist}(v,y) \geq 1$ then x and y have the same color.*

Definition 6. Let C be an ε-coloring.
The *color set* of a point p is the set of colors such that for every $\delta > 0$ there is a point in the δ-neighborhood of p that has this color.
A point is a *k-color point* if its color set contains at least k colors.

Since we may restrict ε-colorings to consist of hexagons, we can avoid k-color points for $k \geq 4$. In contrast to this observation, we can force 3-color points.

Lemma 7. *If a strip has width greater than $\frac{\sqrt{(c-1)^2-1}}{c-1}$, then in every ε-coloring of the strip that uses c colors there is a 3-color point.*

The existance of 3-color points can be exploited to prove upper bounds for $\omega'(c)$.

Theorem 8. $\omega'(4) \leq 1$ *and* $\omega'(5) \leq \sqrt{3}$.

3. Computing $\omega'(4)$

From now on we try to compute $\omega'(4)$. The overall approach is to prove the existance of critical points. These critical points have to fulfill several distance conditions to each other. Each of these conditions can be written as an quadratic inequality in the coordinates of the points. To be able to talk of coordinates we define the lower end of the strip to be the x-axis and choose an orthogonal y-axis. Maximizing the width with respect to these side conditions will provide an upper bound for $\omega'(4)$. On the other hand it will turn out that the solution of this program can be used to construct an ε-coloring of the whole strip.

If the strip has width at least $\frac{\sqrt{8}}{3}$, Lemma 7 yields that there is a 3-color point. The next step is to show that there is a second 3-color point close to the first one.

Lemma 9. *Let C be a ε-coloring with 4 colors of a strip of width $w > \frac{\sqrt{8}}{3}$ and let p be a 3-color point. Then there is a 3-color point q such that $\operatorname{dist}(p,q) < 1$ and the color sets of p and q are distinct.*

For the next steps we need the following assumption.

Assumption 10. *Let C be an ε-coloring with 4 colors of a strip of width $w > \frac{\sqrt{8}}{3}$ and let p be a 3-color point. Then there are no two 3-color points q_1, q_2 such that $\operatorname{dist}(p, q_1) < 1$, $\operatorname{dist}(p, q_2) < 1$ and the color set of p, q_1 and q_2 are pairwise distinct.*

This assumption seems to be sensible since if it would not be true then we get three cycles in different colors. This almost forces the coloring to consist of vertical bars. But this coloring can cover only a strip of width $\frac{\sqrt{8}}{3}$. Every reasonable way to avoid this vertical bar coloring seems to require an even thinner strip.

Lemma 11. *Let p and q be as in* Lemma 9 *and say the color set of p does not contain color 1 and the color set of q does not contain color 2. Then there is a color region of color 1 and a color region of color 2 that have a common border that starts at the upper end of the strip and ends at the lower end of the strip. Furthermore, every part this common border has distance at most 1 to p.*

By observing both sides of this common border, we notice that there are intervals on the lower end and on the upper end of the strip that may only be colored with color 3 and 4.

The common border between the region colored 1 and 2 was forced by two 3-color points. It can be shown that these two points are on the same side of the common border and on the other side of the border there are again two 3-color points with the same color sets. Moreover, on the other side of such a pair of 3-color points there is again a border between regions colored 1 and 2 that starts at the upper end of the strip and ends at the lower end. Hence, this forces a repeating pattern.

Now we can choose some critical points:

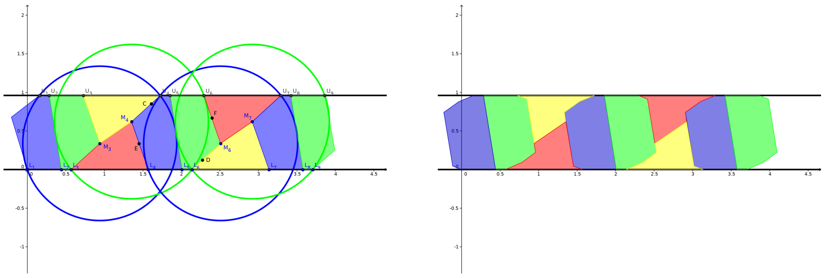
- the 3 color points.
- the points on the ends of the strip of the common border between the regions colored 1 and 2.
- the endpoints of the interval of the union of the regions colored 3 and 4 on the lower end and the upper end of the strip.

Their relations to each other provide several inequalities which are all of a quadratic form:

- The distance between two consequtive points on the upper end of the strip is at most 1.
- The distance between two consequtive points on the lower end of the strip is at most 1.
- The distance between an endpoint of the border and the corresponding endpoints of the intervals is at most 1.
- The distance between an endpoints of the border on the lower end and of the upper end is at most 1.
- The cycle around a 3-color point must not end in the interval that is colored with color 3 and 4 only. So, the distance between a 3-color point to the endpoints of the corresponding intervals is either at most 1 or at least 1 depending on whether it is the start point of the interval or the endpoint.

- The distance between a 3-color point without color 1 and a 3-color point without color 2 on the other side of the border, that is right to the 3-color point without color 1, is at least 2. This has to be ensured to prevent their cycles from intersecting.

Maximizing the width with respect to these inequalities yields a value of $\alpha \approx 0.9588$. If the critical points, for which this optimum is obtained, are simply connected by straight lines then no valid coloring is obtained since the regions colored with color 3 are to close to each other. The same problem appears for the regions colored 4 as in the left picture of Figure 2. This issue can be easily prevented by increasing the size of the regions of colors 1 and 2. This is done by drawing the paralell line to the common border of regions colored 1 and two through the 3-color points and chopping of the parts that would cause a diameter greater than 1 as done in the right picture of Figure 2. The resulting coloring is a valid coloring of a strip of width α and by construction of α there is no coloring of a strip that is any wider.



(a) Straight lines **(b)** Adjusted regions

Figure 2. Result of the quadratic program. The first picture uses straight lines and is therefore not a valid coloring. The second picture adjusts these regions to obtain a valid coloring.

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