TARGET SET IN THRESHOLD MODELS

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ABSTRACT. Consider a graph G and an initial coloring, where each node is blue or red. In each round, all nodes simultaneously update their color based on a predefined rule. In a threshold model, a node becomes blue if a certain number or fraction of its neighbors are blue and red otherwise. What is the minimum number of nodes which must be blue initially so that the whole graph becomes blue eventually? We study this question for graphs which have expansion properties, parameterized by spectral gap, in particular the Erdős–Rényi random graph and random regular graphs.

1. INTRODUCTION

Assume that you are given a graph $G = (V, E)$ and an initial coloring, where each node is either blue or red. In discrete-time rounds, all nodes simultaneously update their color. In the *r-threshold model* for some integer $r \geq 1$, a node becomes blue if it has at least r blue neighbors and red otherwise. In the *α-threshold model* for some $0 < \alpha < 1$, a node becomes blue if at least α fraction of its neighbors are blue and red otherwise. (We assume that r and α are fixed while we let n, the number of nodes in the underlying graph, tend to infinity.)

In each of these two models, a set $T \subseteq V$ is called a *target set* whenever the following holds: if T is fully blue in some round, then the whole graph becomes blue eventually.[1] The minimum size of a target set has been studied extensively on different classes of graphs like lattice, hypercube, random graphs, planar graphs, regular graphs, and many more, cf. [**2, 6, 9, 16, 18, 20**]. We are interested in the graphs with good expansion properties.

There exist different parameters to measure the expansion of a graph. We consider an algebraic characterization of expansion. Assume that $A(G)$ is the adjacency matrix of graph $G = (V, E)$. Consider the normalized adjacency matrix $M = D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$, where D is the diagonal matrix such that the entries of the diagonal are the degrees of the nodes. Let $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -1$ be the eigenvalues of M. We denote the second-largest absolute eigenvalue of the normalized matrix by $\sigma(G) := \max_{2 \leq i \leq n} |\lambda_i|$. Graph G has stronger expansion properties when $\sigma(G)$ is smaller. Here, we assume that $\sigma(G) < 1$.

[1]It is worth to stress that the concept of a target set, known also as *dynamic monopoly*, can be defined more generally to capture similar models like *bootstrap percolation* models, where a blue node remains blue forever, cf. [**15, 7**].

Consider the α-threshold model on a d-regular n-node graph G. Let B_t and R_t denote the set of blue and red nodes in the t-th round and define $b_t := |B_t|$, $r_t := |R_t|$. We prove that if $b_0 \geq \alpha n + \sqrt{2/(1-\alpha)}\,\sigma n$, the graph becomes fully blue and if $b_0 \leq \alpha n - \sqrt{2/\alpha}\,\sigma n$, it becomes fully red in $\mathcal{O}(\log_{1/\sigma^2} n)$ rounds.[2] Roughly speaking, for small values of σ any set "slightly" larger (smaller) than αn is (not) a target set. Assume that initially each node keeps a piece of information, in our case one of the two colors, and we allow an adversary to corrupt almost half of the nodes. Our result asserts that in regular graphs with strong expansion properties if the nodes simply follow the α-threshold model for $\alpha = 1/2$ (that is, each node selects the majority color in its neighborhood, up to the tie-breaking rule), then they all will retrieve the original information in logarithmically many rounds. This is typically known as *density classification property* and has application in distributed fault-local mending where redundant copies of data are kept and the "majority rule" is applied to overcome the damage caused by failures, cf. [16, 17]. We also study the minimum size of a target set in r-threshold model on d-regular graphs with good expansion properties. Moreover, we state that our results can be generalized to include irregular graphs.

From an algorithmic perspective, one might ask what is the minimum size of a target set for a given graph. It is known that this problem is NP-hard in both r-threshold and α-threshold model for certain ranges of α and r, cf. [3, 13, 14]. We consider the same problem for the minimum size of a stable set. It is shown that this problem is NP-hard in α-threshold and r-threshold model for any $0 < \alpha < 1$ and $r \geq 3$. In a graph G, we say a node set S is a *stable set* when the following holds: if S is fully blue in some round, it remains blue forever, regardless of the color of all other nodes. A blue target set results in the full disappearance of red color while a blue stable set only guarantees the survival of blue color.

We present our results regarding the α-threshold and r-threshold model on regular expanders respectively in Sections 2.1 and 2.2. The extension of these results to irregular graphs is given in Section 2.3. Finally, we provide our hardness results concerning the minimum size of a stable set in Section 3.

2. THRESHOLD MODELS ON EXPANDERS

For a node v and a node set S in a graph $G = (V, E)$, define $d_S(v) := |\{u \in S : \{v, u\} \in E\}|$ and let $d(v) = d_V(v)$ be the *degree* of v. We assume that $\delta(G)$ and $\Delta(G)$ denote respectively the minimum and maximum degree in G and define $e(A, B) := |\{(v, u) : v \in A, u \in B, \{v, u\} \in E\}|$. Now, let us provide Lemma 2.1, which is our main tool in this section. Roughly speaking, it states that if σ is small, then the number of edges among every two node sets is almost completely determined by their size.

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[2]This generalizes the results from [19].

Lemma 2.1 (Expander mixing lemma, cf. [**8**])**.** *For a d-regular graph* $G = (V, E)$ *and* $A, B \subset V$

$$|e(A, B) - \frac{|A|\,|B|d}{n}| \leq \sigma d \sqrt{|A|\,|B|\Big(1 - \frac{|A|}{n}\Big)\Big(1 - \frac{|B|}{n}\Big)}.$$

2.1. α-Threshold model

Theorem 2.2. *In the α-threshold model on a d-regular graph $G = (V, E)$,*

(i) *if $b_0 \leq \alpha n - \sqrt{\frac{2}{\alpha}}\,\sigma n$, the graph becomes fully red in $\mathcal{O}(\log_{\frac{\alpha^2}{4\sigma^2}} n)$ rounds,*

(ii) *if $b_0 \geq \alpha n + \sqrt{\frac{2}{1-\alpha}}\,\sigma n$, the graph becomes fully blue in $\mathcal{O}\big(\log_{\frac{(1-\alpha)^2}{4\sigma^2}} n\big)$ rounds.*

Of course, the two aforementioned statements are sensible if respectively $\frac{\alpha^2}{4\sigma^2} > 1$ and $\frac{(1-\alpha)^2}{4\sigma^2} > 1$.

Proof. To prove part (i), let us first show that if $b_0 \leq \alpha n - \sqrt{2/\alpha}\,\sigma n$, then $b_1 \leq \frac{\alpha}{2}n$. Since each node $v \in B_1$ has at least α fraction of its neighbors in set B_0, we have that $e(R_0, B_1) \leq \frac{1-\alpha}{\alpha} e(B_0, B_1)$. Applying Lemma 2.1 yields

$$\frac{r_0 b_1 d}{n} - \sigma d \sqrt{r_0 b_1} \leq \frac{1-\alpha}{\alpha}\Big(\frac{b_0 b_1 d}{n} + \sigma d \sqrt{b_0 b_1}\Big).$$

Multiplying by $\frac{n}{d\sqrt{b_1}}$ and rearranging the terms give us

$$\sqrt{b_1}(r_0 - \frac{1-\alpha}{\alpha}b_0) \leq \sigma n\Big(\sqrt{r_0} + \frac{1-\alpha}{\alpha}\sqrt{b_0}\Big).$$

Since $(\sqrt{r_0} + \frac{1-\alpha}{\alpha}\sqrt{b_0})^2 \leq ((1 + \frac{1-\alpha}{\alpha})\sqrt{n})^2 = \frac{n}{\alpha^2}$ and $r_0 = n - b_0$, we get

$$b_1\Big(n - \Big(\frac{1-\alpha}{\alpha} + 1\Big)b_0\Big)^2 \leq \frac{\sigma^2 n^3}{\alpha^2} \implies b_1 \leq \frac{\sigma^2 n^3}{\alpha^2\big(n - \frac{b_0}{\alpha}\big)^2}.$$

The assumption of $b_0 \leq \alpha n - \sqrt{\frac{2}{\alpha}}\,\sigma n$ implies that $(n - \frac{b_0}{\alpha})^2 \geq (n - n + \sqrt{\frac{2}{\alpha^3}}\,\sigma n)^2 = \frac{2}{\alpha^3}\sigma^2 n^2$. Therefore, we have $b_1 \leq \frac{\sigma^2 n^3}{\alpha^2(2/\alpha^3)\sigma^2 n^2} = \frac{\alpha}{2}n$.

So far we proved that after one round there exist at most $\frac{\alpha}{2}n$ blue nodes. Now, we show that if $b_t \leq \frac{\alpha}{2}n$ for $t \geq 0$, then $b_{t+1} \leq \frac{4\sigma^2}{\alpha^2}b_t$. Thus, the graph becomes fully red in $\mathcal{O}(\log_{\frac{\alpha^2}{4\sigma^2}} n)$ rounds. Since each node in B_{t+1} has at least α fraction of its neighbors in set B_t, we have $\alpha d\,b_{t+1} \leq e(B_t, B_{t+1})$. Applying Lemma 2.1 to the right side yields

$$\alpha d\,b_{t+1} \leq \frac{b_t b_{t+1} d}{n} + \sigma d\,\sqrt{b_t b_{t+1}} \implies \sqrt{b_{t+1}}(\alpha - \frac{b_t}{n}) \leq \sigma\sqrt{b_t}.$$

Utilizing $b_t \leq \frac{\alpha}{2}n$ implies that $b_{t+1} \leq \frac{\sigma^2}{(\alpha-\frac{\alpha}{2})^2}b_t = \frac{4\sigma^2}{\alpha^2}b_t$.

Notice that $b_0 \geq \alpha n + \sqrt{\frac{2}{1-\alpha}}\,\sigma n$ is equivalent to $r_0 \leq (1-\alpha)n - \sqrt{\frac{2}{1-\alpha}}\,\sigma n$. Thus, the proof of part (ii) follows by replacing blue with red and α with $1 - \alpha$ in the above proof. $\square$

Tightness. We show that the logarithmic upper bound in Theorem 2.2 is asymptotically tight; that is, in this setting it might take $\Omega(\log n)$ rounds until the graph is fully red/blue. Assume that $\alpha = 1/2$ and let d be a large constant, say $d \geq 100$. Consider a d-regular graph G with $\sigma \leq 2/\sqrt{d}$, which is known to exist, and even can be constructed explicitly, cf. [**12**]. Assume that initially all nodes are red except the nodes in distance at most $\ell = \frac{1}{2}\log_d n$ from an arbitrary node v. There are at most d^k nodes in distance k from v. Thus, the number of nodes which are blue initially is at most $\sum_{k=0}^{\ell} d^k \leq d^{\ell+1} = \mathcal{O}(\sqrt{n})$. We have that

$$b_0 = \mathcal{O}(\sqrt{n}) \leq \frac{n}{10} \overset{d \geq 100}{\leq} \frac{n}{2} - 2\Big(\frac{2}{\sqrt{d}}\Big) n \overset{\alpha=1/2,\, \sigma \leq 2/\sqrt{d}}{\leq} \alpha n - \sqrt{\frac{2}{\alpha}}\, \sigma n.$$

Thus, by Theorem 2.2 the graph becomes fully red, but it takes at least $\ell = \Omega(\log n)$ rounds since node v will remain red for the first ℓ rounds.

The random d-regular graph $\mathcal{G}_{n,d}$ is the random graph with a uniform distribution over all d-regular graphs on n nodes, say $[n] = \{1, \ldots, n\}$. It is known [**5**] that $\sigma(\mathcal{G}_{n,d}) \leq 2/\sqrt{d}$ for $d \geq 3$ asymptotically almost surely. (We say an event occurs asymptotically almost surely, a.a.s., if it occurs with probability $1 - o(1)$ as n tends to infinity.) Putting this statement in parallel with Theorem 2.2 implies that in the α-threshold model on $\mathcal{G}_{n,d}$ for $d \geq 3$, if $b_0 \leq \alpha n - \sqrt{8/\alpha d}\, n$ then the graph becomes fully red a.a.s. (See [**6**] for similar results on the special case of $\alpha = 1/2$.)

2.2. r-Threshold model

We first provide Lemma 2.3 and Lemma 2.4 about the structure of a regular graph. Building on these two Lemmata, we prove in Theorem 2.5 that in the r-threshold model on a d-regular graph G there is a target set of size at most $2\beta n + 1/\beta$ for $\beta = \frac{r}{(1-\sigma)d}$. (We could apply Theorem 2.2 for $\alpha = r/d$ to find an upper bound of form $(r/d)n + \sqrt{2/(1 - r/d)}\, \sigma n$. However as one can observe, this is a weaker bound for most choices of r, d and σ.)

Lemma 2.3. *In the r-threshold model on a d-regular graph $G = (V, E)$, there is a stable set of size s for $\beta n \leq s \leq 2\beta n + 1/\beta$.*

Proof. Let $G = (V, E)$ be a d-regular graph. We prove that in the r-threshold model on G, there is a stable set of size $\beta n \leq s \leq 2\beta n + 1/\beta$, where $\beta = r/(1-\sigma)d$. Assume that $\mathcal{P}$ is the set of all partitions of V into $\lfloor 1/\beta \rfloor$ sets such that all sets are of size at least $\lfloor \beta n \rfloor$, except one set which can be of size $\lfloor \beta n \rfloor - 1$. Let $P \in \mathcal{P}$ be a partition for which the number of edges between the sets is minimized. Let $V_{\max}$ be a set of maximum size in P. Clearly, $V_{\max}$ is at least of size βn and at most of size

$$n - (\lfloor \tfrac{1}{\beta} \rfloor - 2)\lfloor \beta n \rfloor - (\lfloor \beta n \rfloor - 1) = n - \lfloor \tfrac{1}{\beta} \rfloor \lfloor \beta n \rfloor + \lfloor \beta n \rfloor + 1$$

$$\leq n - \Big(\frac{1}{\beta} - 1\Big)(\beta n - 1) + \beta n + 1$$

which is equal to $2\beta n + 1/\beta$. Therefore, $\beta n \leq |V_{\max}| \leq 2\beta n + 1/\beta$. Furthermore, we claim that for each node $v \in V_{\max}$, $d_{V_{\max}}(v) \geq r$, which implies that $V_{\max}$ is a stable set. Assume that there is a node u which violates this property, i.e., $d_{V_{\max}}(u) \leq r - 1$. Then, the average number of edges between u and the $\lfloor 1/\beta \rfloor - 1$ other sets is at least

$$\frac{d - (r - 1)}{\lfloor 1/\beta \rfloor - 1} \geq \frac{d - (r - 1)}{\frac{(1-\sigma)d}{r} - 1} > \frac{d - (r - 1)}{\frac{d}{r-1} - 1} = r - 1.$$

Thus, there must exist a set V' among the other $\lfloor 1/\beta \rfloor - 1$ sets such that $d_{V'}(u) \geq r$. This is a contradiction because by removing u from $V_{\max}$ and adding it into V', the number of edges between the sets decreases at least by one and it is easy to see that the new partition is also in $\mathcal{P}$. $\square$

Lemma 2.4. *In a d-regular graph $G = (V, E)$, for each set $S \subset V$ of size $s \geq \beta n$, there exists a node $v \in V \smallsetminus S$ so that $d_S(v) \geq r$.*

Proof. By Lemma 2.1, we have

$$\frac{s(n - s)d}{n} - \sigma d \sqrt{s(n - s)\left(\frac{n - s}{n}\right)\left(\frac{s}{n}\right)} \leq e(S, V \smallsetminus S)$$

which yields

$$\frac{s(n - s)(1 - \sigma)d}{n} \leq e(S, V \smallsetminus S).$$

Applying $s \geq \beta n > \frac{(r-1)n}{(1-\sigma)d}$ implies that $(r - 1)(n - s) < e(S, V \smallsetminus S)$. Thus, there exits a node $v \in V \smallsetminus S$ such that $d_S(v) \geq r$ by the pigeonhole principle. $\square$

Theorem 2.5. *In the r-threshold model on a d-regular graph G, there exists a target set of size at most $2\beta n + 1/\beta$.*

Proof. Let S be a stable set of size s such that $\beta n \leq s \leq 2\beta n + 1/\beta$, which must exist by Lemma 2.3. We claim that if initially all nodes in set S are blue, the whole graph becomes blue eventually. Since S is a stable set, all nodes in S stay blue forever. Furthermore by Lemma 2.4, in each round at least one more node becomes blue until the whole graph is blue. (Note that by a simple inductive argument, in each round the nodes in set S and the newly added nodes create a stable set and, thus, remain blue forever.) $\square$

2.3. Irregular graphs

To avoid unnecessary technicalities in the proofs, we limited ourselves to regular graphs so far. Now, we argue that our results from Sections 2.1 and 2.2 can be generalized to capture irregular graphs, by applying basically the same proof ideas. All we need to do is to apply a more general variant of Lemma 2.1, cf. [**8**], and replace d by δ or Δ, according to the case, in the proofs. Then, Theorem 2.2 can be expressed more generally as following. In the α-threshold model on a graph G if $b_0 \leq \underline{b}$, then G becomes fully red in $\mathcal{O}(\log_{\frac{\alpha^2\gamma^2}{4\sigma^2}} n)$ rounds and if $b_0 \geq \overline{b}$, then it

becomes fully blue in $\mathcal{O}(\log_{\frac{(1-\alpha)^2\gamma^2}{4\sigma^2}} n)$ rounds, where $\gamma = \frac{\delta}{\Delta}$

$$\underline{b} = \frac{\gamma^3}{\alpha\gamma^3 + (1-\alpha)}\alpha n - \frac{\sqrt{2/\alpha}}{\alpha\gamma^3 + (1-\alpha)}\sigma n$$

and

$$\overline{b} = \frac{1}{(1-\alpha)\gamma^3 + \alpha}\alpha n + \frac{\sqrt{2/(1-\alpha)}}{(1-\alpha)\gamma^3 + \alpha}\sigma n.$$

Furthermore, in the r-threshold model on a graph G, there is a target set of size at most $2\beta' n + 1/\beta'$, where $\beta' = \frac{r}{(1-\frac{\sigma}{\gamma})\delta}$. Note that in the special case of $\gamma = 1$, this is equivalent to Theorem 2.5 since $\beta' = \beta$.

In the Erdős-Rényi random graph $\mathcal{G}_{n,p}$ each edge is added independently with probability p on a node set of size n, say $[n]$. By the results of Le, Levina, and Vershynin [10], we know that $\sigma(\mathcal{G}_{n,p}) = \mathcal{O}(1/\sqrt{np})$ a.a.s. for $p = \omega(\log n/n)$ (recall that $\log n/n$ is the connectivity threshold). Furthermore, by applying the Chernoff bound [4], $(1-\epsilon)np \leq d(v) \leq (1+\epsilon)np$, for an arbitrarily small constant $\epsilon > 0$, with probability $1 - \exp(-\omega(\log n))$ for an arbitrary node v in $\mathcal{G}_{n,p}$ if $p = \omega(\log n/n)$. The union bound implies that a.a.s. $1 - \epsilon' \leq \gamma$ for an arbitrarily small constant ϵ'. By the results from above, in the α-threshold model on $\mathcal{G}_{n,p}$ with $p = \omega(\log n/n)$, the minimum size of a target set is in $(1 \pm \epsilon'')\alpha n$ a.a.s. for an arbitrarily small constant $\epsilon'' > 0$.

3. Complexity results

Let $MS_\alpha(G)$ and $MS_r(G)$ respectively denote the minimum size of a stable set in α-threshold and r-threshold on a graph G.

Theorem 3.1. *For any constant $0 < \alpha < 1$, the problem of determining $MS_\alpha(G)$ for a given graph G is NP-hard.*

Proof. We provide a reduction from α-CLIQUE, which is the problem of deciding whether a given graph G has a clique of size at least αn or not. It is known that α-CLIQUE, cf. [11], is NP-hard for any constant $0 < \alpha < 1$ by a simple reduction from CLIQUE, which is one of Karp's 21 NP-complete problems.

Let $G = (V = \{v_1, \ldots, v_n\}, E)$ be an instance of α-CLIQUE. We construct a graph $G' = (V', E')$, where $V' := \bigcup_{j=1}^4 V^{(j)} \cup \{w_1, w_2, w_3, w_4\}$ for $V^{(j)} := \{v_i^{(j)} : 1 \leq i \leq n\}$. For the edge set, assume that the induced subgraph $G'[V^{(j)}]$ is an empty graph for $j = 2, 3$ and the induced subgraph $G'[V^{(j)}]$ is a copy of G for $j = 1, 4$. We also connect node w_j to all nodes in $V^{(j)}$ for $1 \leq j \leq 4$. Moreover, we connect nodes $v_i^{(1)}$, $v_{i'}^{(2)}$, and similarly nodes $v_i^{(3)}, v_{i'}^{(4)}$, for $1 \leq i \neq i' \leq n$ if $\{v_i, v_{i'}\} \notin E$. Finally, we add an edge between $v_i^{(2)}, v_{i'}^{(3)}$ for $1 \leq i \neq i' \leq n$ if $\{v_i, v_{i'}\} \in E$.

We claim that G has a clique of size at least αn if and only if $MS_\alpha(G') = \lceil \alpha n \rceil + 1$. Before proving this claim, let us make the following simple, however very useful, observation. A node set S is stable in the α-threshold model if and only if each node v in S has at least $\lceil \alpha d(v) \rceil$ neighbors in S.

The first direction of the claim is quite straightforward. Assume that G has a clique of size at least αn. This implies that it includes a clique of size $\lceil \alpha n \rceil$. There is a clique of the same size in the copy of G on $V^{(1)}$. Let S be the node set obtained by adding w_1 to the node set of this clique. Set S is stable since G' is n-regular by construction and each node $v \in S$ has at least $\lceil \alpha n \rceil$ neighbors in S. Note that by the above observation an n-regular graph cannot have a stable set of size $\lceil \alpha n \rceil$ or smaller. Thus, $MS_\alpha(G') = \lceil \alpha n \rceil + 1$.

Assume that $MS_\alpha(G') = \lceil \alpha n \rceil + 1$. This implies that G' includes a stable set S of size $\lceil \alpha n \rceil + 1$. The induced subgraph by S must be a clique since for S to be stable each node in S has to be connected to $\lceil \alpha n \rceil$ nodes in S. Now, we prove that G includes a clique of size at least αn. Set S includes at most one node from $V^{(2)}$ (similarly $V^{(3)}$) because $G'[V^{(2)}]$ (respectively $G'[V^{(3)}]$) is an empty graph. Furthermore, it is not possible that S has exactly one node $v \in V^{(2)}$ and one node $v' \in V^{(3)}$ because S must include a node, say v'', from $V^{(1)}$ or $V^{(4)}$, which cannot be connected to both v and v'. It is also easy to see that w_2 and w_3 cannot be in S. Therefore by symmetry, we are left with the two following cases: (i) S includes one node from $V^{(2)}$ and the rest of nodes from $V^{(1)}$ or (ii) all nodes from $V^{(1)} \cup \{w_1\}$. Both cases imply that a subset of nodes in $V^{(1)}$ induces a clique of size at least αn; thus, G has also a clique of the same size since $G'[V^{(1)}]$ is a copy of G. $\square$

We observe that in the r-threshold model on a graph $G = (V, E)$, a set S is stable if and only the induced subgraph $G[S]$ has minimum degree at least r. Thus, $MS_1(G) = 2$ if $E \neq \emptyset$. Furthermore, $MS_2(G)$ is equal to the length of the shortest cycle in G, i.e., the girth of G. Therefore, the problem of determining $MS_r(G)$ for a given graph G is in P if $r = 1, 2$. However for $r \geq 3$, the problem is NP-hard.

Theorem 3.2 (Amini, Peleg, Prennes, Sau, and Saurabh [1]). *The problem of determining the minimum size of a set whose induced subgraph has minimum degree at least r in a given graph G does not admit any constant-factor approximation for $r \geq 3$, unless $P = NP$.*

References

1. Amini O., Peleg D., Pérennes S. and Saurabh S., *On the approximability of some degree-constrained subgraph problems*, Discrete Appl. Math. **160** (2012), 1661–1679.
2. Balister P., Bollobás B., Johnson J. R. and Walters M., *Random majority percolation*, Random Structures Algorithms **36** (2010), 315–340.
3. Chen N., *On the approximability of influence in social networks*, SIAM J. Discrete Math. **23** (2009), 1400–1415.
4. Feller W., *An Introduction to Probability Theory and its Applications*, Wiley, New York, 1968.
5. Friedman J., *A proof of alon's second eigenvalue conjecture*, in: Proceedings of the thirty-fifth annual ACM symposium on Theory of computing, 2003, ACM, 720–724.
6. Gärtner B. and Zehmakan A. N., *Majority model on random regular graphs*, in: Proceedings of Latin American Symposium on Theoretical Informatics, Springer, 2018, 572–583.
7. Hambardzumyan L., Hatami H. and Qian Y., *Polynomial method and graph bootstrap percolation*, `arXiv:1708.04640`.
8. Hoory S., Linial N. and Wigderson A., *Expander graphs and their applications*, Bull. Amer. Math. Soc. **43** (2006), 439–561.

9. Jeger C. and N. Zehmakan A., *Dynamic monopolies in two-way bootstrap percolation*, Discrete Appl. Math. **262** (2019), 116–126.

10. Le C. M., Levina E. and Vershynin R., *Concentration and regularization of random graphs*, Random Structures Algorithms **51** (2017), 538–561.

11. Lu Z., Wu W., Chen W., Zhong J., Bi Y. and Gao Z., *The maximum community partition problem in networks*, Discrete Math. Algorithms Appl. **5** (2013), 28–36.

12. Lubotzky A., Phillips R. and Sarnak P. *Ramanujan graphs*, Combinatorica **8** (1998), 261–277.

13. Mishra S., Radhakrishnan J. and Sivasubramanian S., *On the hardness of approximating minimum monopoly problems*, in: Proceedings of International Conference on Foundations of Software Technology and Theoretical Computer Science, Springer, 2002, 277–288.

14. Mishra S. and Rao S., *Minimum monopoly in regular and tree graphs*, Discrete Math. **306** (2006), 1586–1594.

15. Morrison N. and Jonathan A., *Extremal bounds for bootstrap percolation in the hypercube*, J. Combin. Theory Ser. A **156** (156), 61–84.

16. Peleg D., *Local majority voting, small coalitions and controlling monopolies in graphs: A review*, in: Proceedings of 3rd Colloquium on Structural Information and Communication Complexity, 1997, 152–169.

17. Peleg D., *Size bounds for dynamic monopolies*, Discrete Appl. Math. **86** (1998), 263–273.

18. Russo D., Bresler E., Shani U. and Parker J. C., *Dynamic monopolies in tori*, Discrete Appl. Math. **137** (2004), 197–212.

19. Zehmakan A. N., *Opinion forming in Erdös-Rényi random graph and expanders*, in: Proceedings of 29th International Symposium on Algorithms and Computation, 2018, Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.

20. Zehmakan A. N., *Tight bounds on the minimum size of a dynamic monopoly*, in: Proceedings of International Conference on Language and Automata Theory and Applications, Springer, 2019, 381–393.

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