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ON THE ACHIEVABLE AVERAGE DEGREES IN 2-CROSSING-CRITICAL GRAPHS

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Abstract. c-Crossing-critical graphs are the minimal graphs requiring at least c edge crossings in every drawing in the plane. The structure of these obstructions is very rich for every $c \geq 2$. Although, at least in the first nontrivial case of $c = 2$, their structure is well understood. For example, we know that, aside of finitely many small exceptions, the 2-crossing-critical graphs have vertex degrees from the set $\{3, 4, 5, 6\}$ and their average degree can achieve exactly all rational values from the interval $[3\frac{1}{2}, 4\frac{2}{3}]$. Continuing in depth in this research direction, we determine which average degrees of 2-crossing-critical graphs are possible if we restrict their vertex degrees to proper subsets of $\{3, 4, 5, 6\}$. In particular, we identify the (surprising) subcases in which, by number-theoretical reasons, the achievable average degrees form discontinuous sets of rationals.

1. Introduction

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of (pairwise) edge crossings in a drawing of G in the plane. To resolve ambiguity, we consider drawings of graphs such that no edge passes through another vertex and no three edges intersect in a common point which is not their end. A *crossing* is then an intersection point of two edges that is not a vertex.

If a graph has many (e.g., more than linear amount of) edges, then its crossing number must obviously be high. However, even graphs with few edges (e.g., cubic ones) may have quite high crossing number if there is “a lot of nonplanarity” in them. The latter is precisely captured through the critical obstructions: A graph G is *c-crossing-critical* if $\mathrm{cr}(G) \geq c$, but every proper subgraph G' of G has $\mathrm{cr}(G') < c$. Since degree-2 vertices are irrelevant for the crossing number, it is common to assume that crossing-critical graphs have minimum degree 3. Note, though, that crossing-critical graphs may not be simple in general.

There exist only two 1-crossing-critical graphs, K_5 and $K_{3,3}$, but for every $c \geq 2$ there exist an infinite number of c-crossing-critical graphs [**6**]. From Euler's formula (and from the min-degree bound) it is clear that any infinite family of c-crossing-critical graphs, for fixed $c \geq 2$, must have average degree in the interval $[3, 6]$ (it can be shown that the boundary values of 3 and 6 are not achievable,

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but that is nontrivial [**5**]). For any rational $r \in (3, 6)$ and for all sufficiently large integers c there exist infinite families of c-crossing-critical graphs having average degree precisely r [**1**]. On the other hand, the maximum degree in c-crossing-critical graphs is not bounded; it can arbitrarily grow with c [**2**] and even grow for fixed $c \geq 13$ [**3**].

In fact, we nowadays know much more about degree-related properties of infinite families of crossing-critical graphs. In the aforementioned recent paper [**2**] Bokal, Bračič, Derňár, and Hliněný showed that, for sufficiently large c, all the three parameters – crossing number c, rational average degree r, and a set D of degrees that appear arbitrarily often in the graphs of the infinite family – can be prescribed in any reasonable combination. They also analyzed the interplay of the degree parameters specially for 2-crossing-critical graphs (which were recently completely characterized by Bokal, Oporowski, Richter, and Salazar [**4**]).

Although this analysis of degree properties of 2-crossing-critical graphs in [**2**] is fine and very detailed, it has one significant drawback – it has considered only the "frequent" degrees (that occur arbitrarily often) in the families, but it neglected possible other "sporadic" degrees (those with a bounded number of occurrences in the graphs). Hence the interesting question which has remained open since [**2**] is; whether allowing (or forbidding at all) sporadic degrees not from D in the considered infinite critical families makes a negligible or significant difference to the average degree of the family. We completely answer this last question here. In particular, we identify the four surprising cases (of the set D and simplicity of graphs) in which, by number-theoretical reasons, forbidding all degrees not from D severely restricts achievable average degrees compared to the setting of [**2**].

2. Degree-universal critical families

We consider usual graph theory notation, and our graphs may have parallel edges. Since the length of this conference paper is restricted, we can only give some of the core definitions, and we refer the readers to [**2**] for additional concepts and the full context of this research.

Let $D \neq \emptyset$ be a set of integers. A family $\mathcal{G}$ of graphs is called *D-max-universal* if the following holds; (i) for every $d \in D$ and every integer m there is $G \in \mathcal{G}$ such that G contains $\geq m$ vertices of degree d, and (ii) for every $d' \notin D$ there is M such that every $G \in \mathcal{G}$ contains less than M vertices of degree d'. The following has been shown by Bokal, Bračič, Derňár, and Hliněný [**2**]:

Theorem 1 ([**2**]). *A D-max-universal family of 2-crossing-critical graphs exists if and only if $D \subseteq \{3, 4, 5, 6\}$, $|D| \geq 2$, and $D \cap \{3, 4\} \neq \emptyset$. For each such choice of D, the set of achievable average degrees of* D-max-universal *families of 2-crossing-critical graphs (simple or general) is a rational interval listed in* Table 1 *(the right columns).*

A family $\mathcal{G}$ of graphs is *D-perfect-universal* if $\mathcal{G}$ is D-max-universal and every $G \in \mathcal{G}$ has all vertex degrees from D. Our new fine case analysis leads to the following strengthening:

Table 1. The sets of achievable average degrees in D-perfect-universal families of 2-crossing-critical graphs, compared (with highlighted difference) to the D-max-universal setting of [**2**].

	Simple graphs		General graphs	
D	D-perfect-univ.	D-max-univ.	D-perfect-univ.	D-max-univ.
$\{3,4\}$	$\subsetneq [\frac{16}{5}, \frac{18}{5}]$ *	$[\frac{16}{5}, \frac{18}{5}]$	$[\frac{16}{5}, \frac{15}{4}]$	$[\frac{16}{5}, \frac{15}{4}]$
$\{3,5\}$	$\emptyset$	$\{\frac{17}{5}\}$	$\subsetneq [\frac{17}{5}, \frac{11}{3}]$ *	$[\frac{17}{5}, \frac{11}{3}]$
$\{3,6\}$	$\{\frac{18}{5}\}$	$\{\frac{18}{5}\}$	$\{\frac{18}{5}\}$	$\{\frac{18}{5}\}$
$\{4,5\}$	$\emptyset$	$\emptyset$	$\{\frac{9}{2}\}$	$\{\frac{9}{2}\}$
$\{4,6\}$	$\emptyset$	$\emptyset$	$\{\frac{14}{3}\}$	$\{\frac{14}{3}\}$
$\{3,4,5\}$	$(\frac{16}{5}, 4]$	$(\frac{16}{5}, 4]$	$(\frac{16}{5}, \frac{9}{2})$	$(\frac{16}{5}, \frac{9}{2})$
$\{3,4,6\}$	$(\frac{16}{5}, 4)$	$(\frac{16}{5}, 4]$	$(\frac{16}{5}, \frac{14}{3})$	$(\frac{16}{5}, \frac{14}{3})$
$\{3,5,6\}$	$\subsetneq (\frac{17}{5}, \frac{18}{5})$ *	$(\frac{17}{5}, \frac{18}{5})$	$(\frac{17}{5}, \frac{11}{3})$	$(\frac{17}{5}, \frac{11}{3})$
$\{4,5,6\}$	$\emptyset$	$\emptyset$	$\subsetneq (\frac{9}{2}, \frac{14}{3})$ *	$(\frac{9}{2}, \frac{14}{3})$
$\{3,4,5,6\}$	$(\frac{16}{5}, 4]$	$(\frac{16}{5}, 4]$	$(\frac{16}{5}, \frac{14}{3})$	$(\frac{16}{5}, \frac{14}{3})$

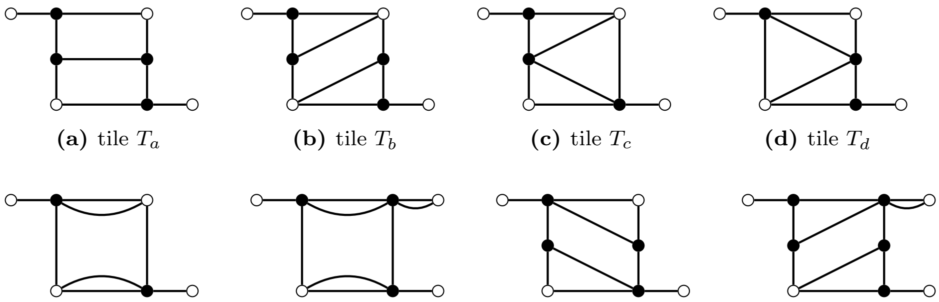
Theorem 2. *For each choice of D as in* Theorem 1*, the set of achievable average degrees of D-*perfect-universal *families of 2-crossing-critical graphs (simple or general) is as listed in* Table 1 *(the left columns). In all cases except the four $*$-marked ones, the achievable set is the whole listed rational interval. In the four $*$-marked cases, the achievable sets are strict subsets of the listed intervals consisting of the relatively prime integer fractions $\frac{p}{q}$ such that, respectively,*

- *for $D = \{3,4\}$ and simple graphs, $p = 2^k x$ and $q = y$ where $k > 0$ and x, y are odd;*
- *for $D = \{3,5,6\}$ and simple graphs, $p = 2^k x$ and $q = y$ where $k > 0$ and x, y are odd;*
- *for $D = \{3,5\}$ (non-simple), $p = 2^k x + y$ and $q = y$ where $k \neq 2$ and x, y are odd;*
- *for $D = \{4,5,6\}$ (non-simple), $p = x$ and $q = 2^k y$, or $p = 2x$ and $q = y$, where $k \geq 0$ and x, y are odd.*

3. Characterization of 2-crossing-critical graphs

Before moving onto the proof of Theorem 2, we have to introduce our main technical tool – the deep characterization result by Bokal, Oporowski, Richter, and Salazar [**4**]. We start with a generic construction of a "twisted band", made of specific elementary pieces called "tiles".

Again, due to a space restriction, we briefly introduce only the core concepts and refer the readers to [**4**] or [**2**] for more details. A *tile* is a plane graph with



(a) tile T_a **(b)** tile T_b **(c)** tile T_c **(d)** tile T_d

(e) tile T_e **(f)** tile T_f **(g)** tile T_g **(h)** tile T_p

Figure 1. Examples of some tiles from the set $\mathfrak{S}$ of Theorem 3 (the names are same as in [**2**]). The vertices of the left and right walls are white (hollow). Notice that each tile has one degree-1 vertex on the left wall and one such on the right wall – this is the same in all tiles from $\mathfrak{S}$.

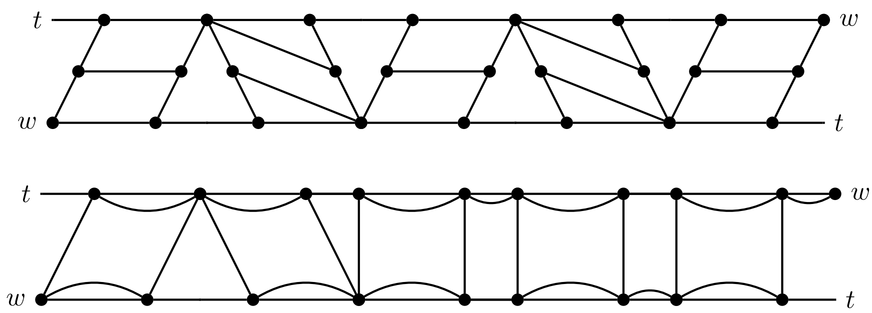


Figure 2. Examples of an alternating join of the following two sequences of tiles: (top) the sequence T_a, T_b, T_a, T_b, T_a, and (bottom) the sequence T_e, T_e, T_f, T_f, T_f. In the corresponding alternating cyclization of each sequence, the two ends marked w are identified together, and likewise the two ends marked t (hence, twisted as in the Möbius band).

an ordered pair of vertices designated as the *left wall* and another pair as the *right wall*, such that both walls lie on the outer face in the respective order. See Figure 1 for an illustration. A *join* of tiles T_1 and T_2 (in this order) is obtained via identification of the right wall of T_1 with the left wall of T_2. Specially, if this identification creates a degree-2 vertex, then it is replaced by a single edge.

The notion of a join is naturally generalized to a sequence of tiles. Having a sequence $T_1, T_2, \dots, T_k$ of tiles, we call an *alternating join* the join of the sequence $T_1, \overline{T}_2, T_3, \overline{T}_4, T_5, \dots, T_k$, where $\overline{T}_j$ denotes the inverted tile of T_j (flipped upside down). Assume now that k is odd, and make the alternating join of $T_1, T_2, \dots, T_k$, followed by identification of the inverted right wall of T_k with the left wall of T_1. The result of this construction is called an *alternating (odd) cyclization* of $T_1, T_2, \dots, T_k$. This is illustrated in Figure 2.

Theorem 3 ([**4**]). *There exist only finitely many 2-crossing-critical graphs which are not 3-connected, and only finitely many of those which are 3-connected and do not contain a subdivision of the graph V_{10} (obtained from a 10-cycle by adding all 5 main diagonals).*

Every 3-connected graph G containing a subdivision of V_{10} is 2-crossing-critical if and only if G is constructed as an alternating cyclization of an odd-length (of at least 3) sequence of tiles from a certain prescribed set $\mathfrak{S}$ of 42 tiles.

4. Proof outline for Theorem 2

First, note that every D-max-universal (so also D-perfect-univ.) family must be infinite from the definition. Hence the finite subcases of Theorem 3 are not relevant for our problem, and we may assume that the considered D-perfect-universal family consists only of alternating odd cyclizations of tiles from $\mathfrak{S}$. In this starting phase our proof is actually quite similar to [**2**].

Consider a tile T with r non-wall vertices and the sum of all degrees s. Let $a = r + 1$ and $b = s - 2$. Then (a, b) is called the *density characteristics* of T and $\frac{b}{a}$ the *density* of T. The reason for this notation is that $r + 1$ and $s - 2$ are the numbers of vertices and edges (resp.) that the tile T contributes to an alternating cyclization, as detailed next.

One can easily verify that if T_i, $i = 0, 1, \ldots, 2k$, are tiles of density characteristics (a_i, b_i), then their alternating cyclization has altogether $n = \sum_{i=0}^{2k} a_i$ vertices, degree sum $m = \sum_{i=0}^{2k} b_i$, and hence the average degree $\frac{m}{n}$. In particular, this means that for studying the achievable average degree in D-perfect-universal families, it is crucial to identify those tiles (from $\mathfrak{S}$) with non-wall degrees from D which have the smallest and the largest densities. (Then, of course, one should also verify that the degrees of vertices created by the join operation are from D.)

In the D-max-universal context of [**2**], the proof of Theorem 1 for each case of D essentially combined the available tiles from $\mathfrak{S}$ with non-wall degrees in D. However, in order to fine-tune the exact average degree and mainly to ensure that the total number of tiles in the resulting cyclization is odd (cf. the condition of Theorem 3 !), [**2**] used to add a special tile (made also of tiles of $\mathfrak{S}$) with degrees possibly outside of D. Since the special tile was of bounded size, it did not affect D-max-universality, but such a trick is now illegal in the D-perfect-universal context.

With respect to previous, we hence structure the proof of Theorem 2 as follows.

I. In the cases in which the degree set D (and a possible restriction to simple graphs) leaves only one possible density or no one at all among the tiles of $\mathfrak{S}$, we can use the same proof as in [**2**] and get the same result. In Table 1, those are the entries having $\emptyset$ or a single value, except the row $D = \{3, 5\}$.

II. In the cases in which the restriction to the degree set D allows at least three different tile densities among $\mathfrak{S}$, we suitably combine copies of several of the available tiles (usually of three, but up to five in a case) to achieve at the same time the desired average degree and an odd total number of

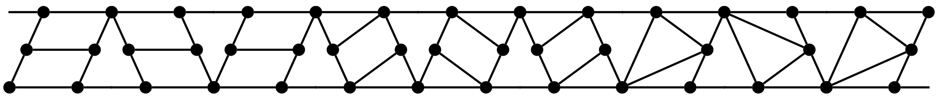


Figure 3. An example of the alternating join of an odd-length sequence from tiles T_a, T_g and T_c producing only degrees 3, 4 and 5. These tiles have (in order) the minimum density $\frac{16}{5}$, the maximum density $\frac{16}{4}$ and an "intermediate" density $\frac{18}{5}$.

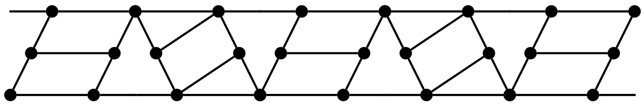


Figure 4. An example of the alternating join of an odd-length sequence from simple tiles T_a and T_g producing only degrees 3,4. This restriction on the set of degrees does not allow any other tile density to occur, and hence some average degrees are not achievable with odd-length sequences.

tiles. This leads to constructions of D-perfect-universal families with every rational average degree between the minimal and the maximal available densities, and the availability of a third "intermediate" density value is crucial to overcome the associated number-theoretical obstacles.

Since this part is routine and conceptually similar to previous [**2**], we only refer to the Bachelor's thesis [**7**] of the second author for all details, and present an example in Figure 3.

III. In addition to the previous, we specifically mention the simple-graph cases of (a) $D = \{3, 5\}$ and density $\frac{17}{5}$, and (b) $D = \{3, 4, 6\}$ and density 4 (two of the six highlighted differences in Table 1). In each exactly two tiles are available – T_a, T_b in (a) and T_c, T_d in (b) (see Fig. 1), and they have to alternate in a sequence in order not to create a vertex of a forbidden degree, such as the degree-4 vertex w in the top example of Figure 2. Consequently, we get no odd-length sequence for an admissible cyclization and the densities are not achievable.

IV. We are left with the four $*$-marked cases in Table 1. In each of them, [**2**] have shown that there exist only two admissible tile densities (the lower and upper extremes of the respective density interval $\mathcal{J}$ from the table) in $\mathfrak{S}$, and they have constructed examples for each average degree from $\mathcal{J}$. Now we have, by Theorem 2, only a discontinuous subset $\mathcal{I} \subsetneq \mathcal{J}$ of achievable average degrees of D-perfect-universal families.

Similarly as in (II.), we can always suitably combine an odd number of copies of the two available tiles to achieve the resulting average degree from $\mathcal{I}$, e.g., as in the example in Figure 4.

The more interesting part is to prove that average degrees in $\mathcal{J} \setminus \mathcal{I}$ are not achievable. We use the following claim:

– Let T_1, T_2 be tiles of densities $\frac{b_1}{a_1}$, $\frac{b_2}{a_2}$, and let $\frac{b_1}{a_1} < r < \frac{b_2}{a_2}$. Then every sequence of copies of T_1, T_2 achieving average degree r after cyclization has length which is a multiple of the length of a minimal such sequence.

The way we apply this claim is to constructively show that, for each average degree $r \in \mathcal{J} \setminus \mathcal{I}$, there exists such a minimal sequence of even length, and so odd-length sequences are not possible. $\square$

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