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A NOTE ON COVERING YOUNG DIAGRAMS WITH APPLICATIONS TO LOCAL DIMENSION OF POSETS

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ABSTRACT. We prove that in every cover of a Young diagram with $\binom{2k}{k}$ steps with generalized rectangles there is a row or a column in the diagram that is used by at least $k+1$ rectangles. We show that this is best-possible by partitioning any Young diagram with $\binom{2k}{k}-1$ steps into actual rectangles, each row and each column used by at most k rectangles. This answers two questions by Kim et al. [**2**].

Our results can be rephrased in terms of local covering numbers of difference graphs with complete bipartite graphs, which has applications in the recent notion of local dimension of partially ordered sets.

1. Introduction

A *Young diagram* with r rows and c columns is a subset $Y \subseteq [r] \times [c]$ such that whenever $(i,j) \in Y$, then $(i-1,j) \in Y$ provided $i \geq 2$, as well as $(i,j-1) \in Y$ provided $j \geq 2$. A Young diagram is visualized as a set of unit squares that are arranged consecutively in rows and columns, each row starting in the first column, and with every row (except the first) being at most as long as the row above. The number of steps of a Young diagram Y is the number of different row lengths in Y, i.e., the cardinality of $Z = \{(s,t) \in Y \mid (s+1,t) \notin Y \text{ and } (s,t+1) \notin Y\}$. The elements of Z are called *steps* of Y. Young diagrams with n elements, r rows, c columns, and z steps, visualize integer partitions of n into r parts of size $\leq c$ using z different sizes. In the literature our Young diagrams are more frequently called Ferrers diagrams. We stick to Young diagram to be consistent with [**2**].

A *generalized rectangle* in a Young diagram $Y \subseteq [r] \times [c]$ is a set R of the form $R = S \times T$ with $S \subseteq [r]$ and $T \subseteq [c]$ and $R \subseteq Y$. Note that not every set of the form $R = S \times T$ with $S \subseteq [r]$ and $T \subseteq [c]$ satisfies $R \subseteq Y$ (unless $Y = [r] \times [c]$). A generalized rectangle $R = S \times T$ with S being a set of consecutive numbers in $[r]$ and T being a set of consecutive numbers in $[c]$ is an *actual rectangle*. A generalized rectangle $R = S \times T$ *uses* the rows in S and the columns in T. See the left of Figure 1 for an illustrative example.

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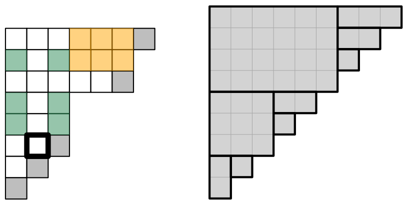

Figure 1. Left: A Young diagram Y with $r=8$ rows, $c=7$ columns, and $z=5$ steps. Highlighted are the set Z of steps (gray), the element $(i,j)=(6,2)\in Y$ (bold boundary), the generalized rectangle $\{2,4,5\}\times\{1,3\}$ (green), and the actual rectangle $\{1,2\}\times\{4,5,6\}$ (orange). **Right:** The Young diagram Y_9 with 9 steps and a $(2,3)$-local partition of Y with actual rectangles.

Motivated by applications for the local dimension of partially ordered sets, we investigate covering a Young diagram Y with generalized rectangles such that every row and every column of Y is used by as few generalized rectangles in the cover as possible. We say that Y is *covered* by a set C of generalized rectangles if $Y=\bigcup_{R\in C} R$, i.e., Y is the union of all rectangles in C. In this case we also say that C is a *cover* of Y. If additionally the rectangles in C are pairwise disjoint, we call C a *partition* of Y. For example, the right of Figure 1 shows a Young diagram with a partition into actual rectangles.

Theorem 1. *For any $k\in\mathbb{N}$, any Young diagram Y can be covered by a set C of generalized rectangles such that each row and each column of Y is used by at most k rectangles in C if and only if Y has strictly less than $\binom{2k}{k}$ steps.*

We prove Theorem 1 in Section 2, answer the questions raised by Kim et al. in Section 3, and describe the application to local dimension of posets in Section 4.

2. Proof of Theorem 1

We use the term *rectangle* for generalized rectangles, and the term *actual rectangle* for rectangles that are contiguous. For a Young diagram Y and $i,j\in\mathbb{N}$, let us define a cover C of Y to be (i,j)*-local* if each row of Y is used by at most i rectangles in C and each column of Y is used by at most j rectangles in C. For $z\in\mathbb{N}$, let $Y_z=\{(s,t)\in[z]\times[z]\mid s+t\le z+1\}$ be the (unique) Young diagram with z rows, z columns, and z steps. See the right of Figure 1.

We start with a lemma stating that instead of considering any Young diagram with z steps, we may restrict our attention to just Y_z.

Lemma 2. *Let $i,j,z\in\mathbb{N}$ and Y be any Young diagram with z steps. Then Y admits an (i,j)-local cover if and only if Y_z admits an (i,j)-local cover with exactly z rectangles.*

We omit the proof and refer to Figure 2 for an illustration of the proof idea.

Let us now turn to our main result. In fact, we shall prove the following strengthening of Theorem 1.

Theorem 3. *For any $i,j,z\in\mathbb{N}$ and any Young diagram Y with z steps, the following hold.*

(i) *If $z<\binom{i+j}{i}$, then there exists an (i,j)-local partition of Y with actual rectangles.*

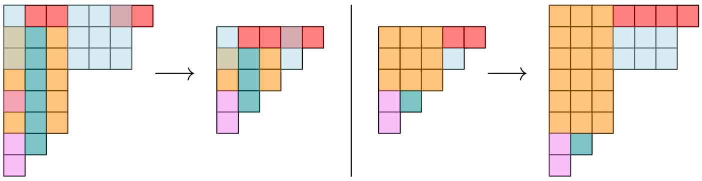

Figure 2. Transforming a cover of any Young diagram Y with 5 steps into a cover of Y_5 (left) and vice versa (right).

(ii) *If $z \geq \binom{i+j}{i}$, then there exists no (i,j)-local cover of Y with generalized rectangles.*

Proof. We define $f(i,j) := \binom{i+j}{i} - 1$. It is crucial that the numbers $\{f(i,j)\}_{i,j\geq 1}$ satisfy the recursion $f(i,j) = f(i-1,j) + f(i,j-1) + 1$ for $i,j \geq 2$ with initial conditions $f(1,j) = j = f(j,1)$. Due to Lemma 2 it suffices to show that for any $i,j \in \mathbb{N}$ and $z = f(i,j) = \binom{i+j}{i} - 1$, there is an (i,j)-local partition of Y_z with actual rectangles.

We omit the proof of (i) and refer to Figure 3 for an illustration of the contruction of a (i,j)-local partition of Y_z with $z \leq f(i,j)$.

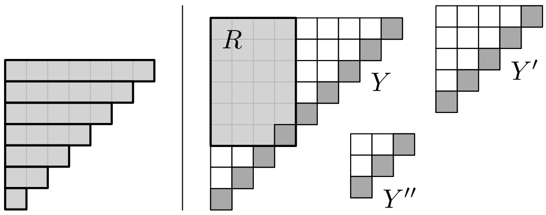

Figure 3. Left: The Young diagram Y_z with $z = f(1,7) = \binom{1+7}{1} - 1 = 7$ steps and a $(1,7)$-local partition of Y_z into actual rectangles. **Right:** The Young diagram Y_z with $z = f(3,2) = \binom{3+2}{3} - 1 = 9$ steps, the rectangle $R = [a] \times [z+1-a] = [6] \times [4]$ with $a = f(2,2) + 1 = 6$, and the Young diagrams Y' and Y'' with $f(2,2) = 5$ and $f(3,1) = 3$ steps, respectively.

We provide the details for the proof of (ii). Due to Lemma 2 it is sufficient to show that for $i,j \in \mathbb{N}$ the Young diagram $Y_{z'}$ with $z' \geq \binom{i+j}{i}$ admits no (i,j)-local cover. If $Y_{z'}$ with $z' > z = \binom{i+j}{i}$ has an (i,j)-local cover, then by restricting the rectangles of the cover to the rows from $z'-z$ to z' we obtain an (i,j)-local cover of a down-shifted copy of Y_z. Therefore, we only have to consider Y_z.

Let C be a cover of Y_z. We shall prove that C is not (i,j)-local. Again, we proceed by induction on i and j, where illustrations are given in Figure 4.

If $i = 1$, then each row is only used by a single rectangle in C. Hence, each row of Y_z is a rectangle in C. Thus column 1 of Y_z is used by $z = j+1$ rectangles, proving that C is not (i,j)-local. The case $j = 1$ is alike.

Now let $i \geq 2$ and $j \geq 2$. We have $z = \binom{i+j}{i} = \binom{(i-1)+j}{i-1} + \binom{i+(j-1)}{i}$. Consider the rectangle $M = [a] \times [z-a]$ for $a = \binom{(i-1)+j}{i-1}$. Then $Y_z - M$ splits into a right-shifted Y' copy of Y_a and a down-shifted copy Y'' of Y_{z-a}.

Let C', respectively C'', be the subset of rectangles in C using at least one of the rows $1,\ldots,a$ in Y_z, respectively at least one of the columns $1,\ldots,z-a$ in Y_z. Note that $C' \cap C'' = \emptyset$ as each rectangle of C is contained in Y_z. Prune rectangles in C' and C'' to obtain covers of Y' and Y''.

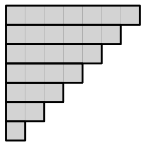

Figure 4. The Young diagram Y_z with $z = \binom{3+2}{3} = 10$ steps, the rectangle $M = [a] \times [z-a] = [6] \times [4]$ with $a = \binom{2+2}{2} = 6$, and the Young diagrams Y' and Y'' with $\binom{2+2}{2} = 6$ and $\binom{3+1}{3} = 4$ steps, respectively.

The Young diagram Y' is a copy of Y_a and $a = \binom{(i-1)+j}{i-1}$. Hence, by induction the pruned cover C' is not $(i-1, j)$-local. If some column t of Y' is used by at least $j+1$ rectangles in C', this column of Y_z is used by at least $j+1$ rectangles in C, proving that C is not (i,j)-local, as desired. So we may assume that some row s of Y' is used by at least i rectangles in C'.

Symmetrically, Y'' is a copy of Y_{z-a} and $z-a = \binom{i+(j-1)}{i}$. Hence, the pruned C'' is a cover of Y'', which by induction is not $(i, j-1)$-local, and we may assume that some column t of Y'' is used by at least j rectangles in C''. Hence row s in Y_z is used by at least i rectangles in C' and column t in Y_z is used by at least j rectangles in C''. As $C' \cap C'' = \emptyset$ and element (s,t) is contained in some rectangle of C, either row s of Y_z is used by at least $i+1$ rectangles or column t of Y_z is used by at least $j+1$ rectangles (or both), proving that C is not (i,j)-local. $\square$

3. Local covering numbers

A *difference graph* is a bipartite graph in which the vertices of one partite set can be ordered $a_1, \ldots, a_r$ in such a way that $N(a_i) \subseteq N(a_{i-1})$ for $i = 2, \ldots, r$, i.e., the neighborhoods of these vertices along this ordering are weakly nesting. Equivalently, a bipartite graph H is a difference graph iff H admits a bipartite adjacency matrix whose support is a Young diagram Y. Complete bipartite subgraphs H correspond to generalized rectangles in Y. In [**2**], Kim et al. investigate the relations between *local difference cover numbers* and *local complete bipartite cover numbers* as described below[1].

Following the notation in [**3**], local covering numbers are defined as follows. For a graph class $\mathfrak{F}$ and a graph H, an *injective $\mathfrak{F}$-covering* of H is a set of graphs $G_1, \ldots, G_t \in \mathfrak{F}$ with $H = G_1 \cup \cdots \cup G_t$. An injective $\mathfrak{F}$-covering of H is *k-local* if every vertex of H is contained in at most k of the graphs $G_1, \ldots, G_t$, and the *local $\mathfrak{F}$-covering number* of H, denoted by $\mathrm{c}_\ell^{\mathfrak{F}}(H)$, is the smallest k for which a k-local injective $\mathfrak{F}$-cover of H exists.

Let $\mathfrak{D}$ denote the class of all difference graphs, and $\mathfrak{CB} \subset \mathfrak{D}$ the class of all complete bipartite graphs. Clearly, we have $\mathrm{c}_\ell^{\mathfrak{D}}(H) \leq \mathrm{c}_\ell^{\mathfrak{CB}}(H)$ for all graphs H. Kim et al. [**2**] prove that $\mathrm{c}_\ell^{\mathfrak{CB}}(H) \leq \mathrm{c}_\ell^{\mathfrak{D}}(H) \cdot \lceil \log_2(n/2+1) \rceil$ for all H on n vertices, by showing that $\mathrm{c}_\ell^{\mathfrak{CB}}(H) \leq \lceil \log_2(r+1) \rceil$ whenever $H \in \mathfrak{D}$ is a difference graph with one partite set of size r. However, no lower bound on $\mathrm{c}_\ell^{\mathfrak{CB}}(H)$ for $H \in \mathfrak{D}$ is established in [**2**]. Specifically, Kim et al. ask for the exact value of $\mathrm{c}_\ell^{\mathfrak{CB}}(H_i)$ for

[1] Deviating from [**2**], we follow here the terminology and notation of local covering numbers introduced in [**3**].

the difference graph H_i corresponding to the Young diagram Y_i. For the case that $i+1$ is a power of 2 they prove the upper bound $\mathrm{c}_\ell^{\mathfrak{CB}}(H_i) \leq \log_2(i+1) - 1$.

Using Theorem 1 and $\binom{2k}{k} = (1+o(1))\frac{1}{\sqrt{k\pi}}2^{2k}$, we obtain:
1) For every difference graph H the exact value of $\mathrm{c}_\ell^{\mathfrak{CB}}(H)$ is the smallest $k \in \mathbb{N}$ such that for the number z of steps of the Young diagram Y_H of H it holds $z < \binom{2k}{k}$.
2) The difference graphs H_i of Kim et al. satisfy $\mathrm{c}_\ell^{\mathfrak{CB}}(H_i) = (1+o(1))\frac{1}{2}\log_2 i$.
3) For all graphs H on n vertices, $\mathrm{c}_\ell^{\mathfrak{CB}}(H) \leq \mathrm{c}_\ell^{\mathfrak{D}}(H) \cdot (1+o(1))\frac{1}{2}\log_2(n/2)$.

4. Local dimension of posets

The motivation for Kim et al. [**2**] to study local difference cover numbers comes from the local dimension of posets, a notion recently introduced by Ueckerdt [**4**].

Define a *partial linear extension* of poset $\mathcal{P}$ to be a linear extension L of an induced subposet of $\mathcal{P}$. A *local realizer* of $\mathcal{P}$ is a non-empty set $\mathcal{L}$ of partial linear extensions such that **(1)** if $x < y$ in $\mathcal{P}$, then $x < y$ in some $L \in \mathcal{L}$, and **(2)** if x and y are incomparable (denoted $x||y$), then $x < y$ in some $L \in \mathcal{L}$ and $y < x$ in some $L' \in \mathcal{L}$. The *local dimension* of $\mathcal{P}$, denoted $\mathrm{ldim}(\mathcal{P})$, is the smallest k for which there exists a local realizer $\mathcal{L}$ of $\mathcal{P}$ with each $x \in P$ appearing in at most k partial linear extensions $L \in \mathcal{L}$.

For an arbitrary height-two poset $\mathcal{P} = (P, \leq)$, Kim et al. consider the bipartite graph $G_\mathcal{P} = (P, E)$ with partite sets $A = \min(\mathcal{P})$ and $B = P - \min(\mathcal{P}) \subseteq \max(\mathcal{P})$ whose edges correspond to the incomparable A, B pairs (a.k.a. critical pairs. They prove that $\mathrm{c}_\ell^{\mathfrak{D}}(G_\mathcal{P}) - 2 \leq \mathrm{ldim}(\mathcal{P}) \leq \mathrm{c}_\ell^{\mathfrak{CB}}(G_\mathcal{P}) + 2$, which also gives good bounds for $\mathrm{ldim}(\mathcal{P})$ when $\mathcal{P}$ has larger height, since if $\mathcal{Q}$ is the split of $\mathcal{P}$, then $\mathrm{ldim}(\mathcal{Q}) - 2 \leq \mathrm{ldim}(\mathcal{P}) \leq 2\,\mathrm{ldim}(\mathcal{Q}) - 1$ (see [**1**], Lemma 5.5). Using these results and the ones from the previous section, we can conclude the following for the local dimension of any poset: $\mathrm{c}_\ell^{\mathfrak{D}}(G_\mathcal{Q}) - 4 \leq \mathrm{ldim}(\mathcal{P}) \leq \mathrm{c}_\ell^{\mathfrak{D}}(G_\mathcal{Q}) \cdot (1+o(1))\log_2 n$.

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After publishing our note on arXiv we were contacted by two other groups which have independently proved Theorem 3, these are Balázs Keszegh, Dániel T. Nagy, Gábor Damásdi (Budapest) and António Girão, David Lewis (Memphis).

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