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CLASSIFICATIONS OF $N(k)$-CONTACT METRIC MANIFOLDS SATISFYING CERTAIN CURVATURE CONDITIONS

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Abstract. The object of the present paper is to classify $N(k)$-contact metric manifolds satisfying certain curvature conditions on the projective curvature tensor. Projectively pseudosymmetric and pseudoprojectively flat $N(k)$-contact metric manifolds are considered. Beside these we also study $N(k)$-contact metric manifolds satisfying $\widetilde{Z} \cdot P = 0$, where $\widetilde{Z}$ and P denote, respectively, the concircular and projective curvature tensor, respectively.

1. Introduction

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well-known projective curvature tensor P vanishes. Here P is defined by [**20**]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y] \tag{1}$$

for all X, Y, $Z \in T(M)$, where R is the curvature tensor and S is the Ricci tensor. In fact, M is projectively flat if and only if it is of constant curvature [**24**]. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature. Let (M,g) be a Riemannian manifold and let ∇ be the Levi-Civita connection of (M,g). A Riemannian manifold is called locally symmetric [**9**] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M,g). A Riemannian manifold M is called semisymmetric if

$$R \cdot R = 0 \tag{2}$$

holds, where R denotes the curvature tensor of the manifold. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. Semisymmetric Riemannian manifolds

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were first studied by E. Cartan, A. Lichnerowich, R. S. Couty and N. S. Sinjukov. A fundamental study on Riemannian semisymmetric manifolds was made by Z. I. Szabó [**21**], E. Boeckx et al [**7**] and O. Kowalski [**15**]. A Riemannian manifold M is said to be Ricci-semisymmetric if on M we have

$$R \cdot S = 0, \tag{3}$$

where S is the Ricci tensor.

The class of Ricci-semisymmetric manifolds includes the set of Ricci-symmetric manifolds ($\nabla S = 0$) as a proper subset. Ricci-semisymmetric manifolds were investigated by several authors. We define the subsets U_R, U_S of a Riemannian manifold M by $U_R = \{x \in M : R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x\}$ and $U_S = \{x \in M : S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$, respectively, where $G(X,Y)Z = g(Y,Z)X - g(X,Z)Y$. Evidently, we have $U_S \subset U_R$. A Riemannian manifold is said to be pseudo-symmetric [**23**] if at every point of M the tensor $R \cdot R$ and $Q(g,R)$ are linearly dependent. This is equivalent to

$$R \cdot R = f_R Q(g,R)$$

on U_R, where f_R is a function on U_R. Clearly, every semi-symmetric manifold is pseudo-symmetric but the converse is not true [**23**].

A Riemannian manifold M is said to be Ricci pseudo-symmetric if $R \cdot S$ and $Q(g,S)$ on M are linearly dependent. This is equivalent to

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that holds on U_S, where f_S is a function defined on U_S.

In [**6**], Blair et al. studied $N(k)$-contact metric manifold satisfying the curvature conditions $\widetilde{Z} \cdot \widetilde{Z} = 0$, $\widetilde{Z} \cdot R = 0$ and $R \cdot \widetilde{Z} = 0$, where $\widetilde{Z}$ is the concircular curvature tensor ([**24**], [**25**]) defined by

$$\widetilde{Z}(X,Y)W = R(X,Y)W - \frac{r}{2n(2n+1)}[g(Y,W)X - g(X,W)Y], \tag{4}$$

where $X, Y, W \in TM$ and r is the scalar curvature. Recently, De et al. [**10**] studied $N(k)$-contact metric manifolds satisfying the curvature conditions $P \cdot R = 0$, $P \cdot S = 0$ and $P \cdot P = 0$. Motivated by the above studies, we characterize $N(k)$-contact metric manifolds satisfying certain curvature conditions on the projective curvature tensor. The paper is organized as follows.

In this paper, we study projective curvature tensor on $N(k)$-contact metric manifolds. After Preliminaries in Section 3, we consider projectively pseudosymmetric $N(k)$-contact manifolds. Section 4 deals with the study of pseudoprojectively flat $N(k)$-contact metric manifolds. Section 4 is devoted to study $N(k)$-contact metric manifolds satisfying $\widetilde{Z} \cdot P = 0$.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form

η satisfying ([**2**],[**3**])

$$(5) \quad (a)\ \phi^2 = -I + \eta \otimes \xi, \quad (b)\ \eta(\xi) = 1, \quad (c)\ \phi\xi = 0, \quad (d)\ \eta \circ \phi = 0.$$

An almost contact structure is said to be normal if the almost complex structure J on the product manifold defined by

$$J\Big(X, f\frac{\mathrm{d}}{\mathrm{d}t}\Big) = \Big(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\Big)$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M\times\mathbb{R}$. Let g be a compatible Riemannian metric with almost contact metric structure (ϕ, η, ξ), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{6}$$

Then M becomes an almost contact metric structure (ϕ, ξ, η, g). From (5), it can be easily seen that

$$(7) \qquad (a)\ \ g(X, \phi Y) = -g(\phi X, Y), \qquad (b)\ \ g(X, \xi) = \eta(X)$$

for all vector fields X, Y. An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = \mathrm{d}\eta(X, Y) \tag{8}$$

for all vectors fields X, Y. The 1-form η is called a contact metric form and ξ is its characteristic vector field. We define a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi\phi$, where $\pounds$ denotes the Lie derivative. Then h is symmetric and satisfies the conditions $h\phi = -\phi h$, $Tr \cdot h = Tr \cdot \phi h = 0$ and $h\xi = 0$. Also

$$\nabla_X\xi = -\phi X - \phi h X, \tag{9}$$

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is a Sasakian manifold if and only if

$$(\nabla_X\phi)(Y) = g(X, Y)\xi - \eta(Y)X, \tag{10}$$

where $X, Y \in TM$ and ∇ is the Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is a Killing vector field is said to be a K-contact metric manifold. A Sasakian manifold is K-contact but not conversely. However, a 3-dimensional K-contact metric manifold is Sasakian [**14**]. It is known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$ [**4**]. On the other hand, on a Sasakian manifold, the following relation

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{11}$$

holds.

As a generalization of both $R(X, Y)\xi = 0$ and the Sasakian case, D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [**5**] introduced the (k, μ)-nullity distribution on a contact metric manifold and gave several reasons for studying it. The

(k,μ)-nullity distribution $N(k,\mu)$ ([**5**], [**18**]) of a contact metric manifold M is defined by

$$\begin{aligned}&N(k,\mu)\colon p\to N_p(k,\mu)\\&=\{W\in T_pM : R(X,Y)W=(kI+\mu h)(g(Y,W)X-g(X,W)Y)\}\end{aligned}$$

for all $X,Y\in TM$, where $(k,\mu)\in\mathbb{R}^2$. A contact metric manifold M with $\xi\in N(k,\mu)$ is called a (k,μ)-contact metric manifold. The (k,μ)-contact metric manifolds were studied by Papantonious [**18**], De and Sarkar [**11**], Özgür ([**1**], [**16**]) and many others. If $\mu=0$, the (k,μ)-nullity distribution reduces to k-nullity distribution [**22**]. The k-nullity distribution $N(k)$ of a Riemannian manifold is defined by [**22**]

$$N(k)\colon p\to N_p(k)=\{Z\in T_pM : R(X,Y)Z=k[g(Y,Z)X-g(X,Z)Y]\},$$

k being a constant. If the characteristic vector field $\xi\in N(k)$, then we call a contact metric manifold as an $N(k)$-contact metric manifold [**6**]. The $N(k)$-contact metric manifolds were studied by Blair et al. [**6**], De et al. ([**10**], [**12**], [**13**]), Sing et al. [**19**], Özgür et al. [**17**] and many others.

However, for an $N(k)$-contact metric manifold M of dimension $(2n+1)$, we have [**6**]

$$(\nabla_X\phi)Y=g(X+hX,Y)\xi-\eta(Y)(X+hX), \tag{12}$$

where $h=\frac{1}{2}\pounds_\xi\phi$,

$$h^2=(k-1)\phi^2, \tag{13}$$

$$R(X,Y)\xi=k[\eta(Y)X-\eta(X)Y], \tag{14}$$

$$\begin{aligned}S(X,Y)&=2(n-1)g(X,Y)+2(n-1)g(hX,Y)\\&\quad+[2nk-2(n-1)]\eta(X)\eta(Y),\qquad n\geq 1.\end{aligned} \tag{15}$$

$$S(Y,\xi)=2nk\eta(X), \tag{16}$$

$$(\nabla_X\eta)(Y)=g(X+hX,\phi Y), \tag{17}$$

$$(\nabla_Xh)(Y)=\{(1-k)g(X,\phi Y)+g(X,h\phi Y)\}\xi+\eta(Y)[h(\phi X+\phi hX)] \tag{18}$$

for any vector fields X,Y,Z, where R is the Riemannian curvature tensor and S is the Ricci tensor.

In [**4**] Blair proved the following theorem.

Theorem 2.1 ([**4**])**.** *A $(2n+1)$ dimensional contact metric manifold satisfying $R(X,Y)\xi=0$ is locally isometric to $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for $n=1$.*

We also recall the notion of D-homothetic deformation. For a given contact metric structure (ϕ,ξ,η,g), this is the structure defined by

$$\bar\eta=a\eta,\qquad \bar\xi=\frac{1}{a}\xi,\qquad \bar\phi=\phi,\qquad \bar g=ag+a(a-1)\eta\otimes\eta,$$

where a is a positive constant. While such a change preserves the state of being contact metric, K-contact, Sasakian or strongly pseudo-convex CR, it destroys a condition as $R(X,Y)\xi = 0$ or $R(X,Y)\xi = k\{\eta(Y)X - \eta(X)Y\}$.
However, the form of the (k,μ)-nullity condition is preserved under a D-homothetic deformation with

$$\bar{k} = \frac{k + a^2 - 1}{a^2}, \qquad \bar{\mu} = \frac{\mu + 2a - 2}{a},$$

Given a non-Sasakian (k,μ)-contact manifold M, Boeckx [**8**] introduced an invariant

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1-k}}$$

and showed that for two non-Sasakian (k,μ)-manifolds $(M_i, \phi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$, we have $I_{M_1} = I_{M_2}$ if and only if up to a D-homothetic deformation, the two manifolds are locally isometric as contact metric manifolds.
Thus we see that from all non-Sasakian (k,μ)-manifolds locally as soon as we have for every odd dimension $(2n+1)$ and for every possible value of the invariant I, one (k,μ) manifold M can be obtained with $I_M = I$. For $I > -1$, such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c where we have $I = \frac{1+c}{|1-c|}$. Boeckx also gives a Lie algebra construction for any odd dimension and value of $I \leq -1$.

Example 2.1. [**6**]

Using this invariant, Blair et al. [**6**] constructed an example of a $(2n+1)$-dimensional $N(1-\frac{1}{n})$-contact metric manifold, $n > 1$. The example is given as follows: Since the Boeckx invariant for a $(1-\frac{1}{n}, 0)$-manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n+1)$-dimensional manifold of constant curvature c such that the resulting D-homothetic deformation is a $(1-\frac{1}{n}, 0)$-manifold. That is, for $k = c(2-c)$ and $\mu = -2c$, we solve

$$1 - \frac{1}{n} = \frac{k + a^2 - 1}{a^2}, \qquad 0 = \frac{\mu + 2a - 2}{a}$$

for a and c. The result is

$$c = \frac{\sqrt{n} \pm 1}{n-1}, \qquad a = 1 + c,$$

and taking c and a to be these values, we obtain an $N(1-\frac{1}{n})$-contact metric manifold.
The above example will be used in Theorem 5.1.

3. Projectively pseudosymmetric $N(k)$-contact metric manifolds

A Riemannian manifold is said to be projectively pseudosymmetric [**23**] if at every point of the manifold the following relation

$$(R(X,Y) \cdot R)(U,V)W = L_R((X \wedge Y) \cdot R)(U,V)W) \tag{19}$$

holds for any vector fields X, Y, U, V, $W \in TM$, where L_R is an function of M. The endomorphism $X \wedge Y$ is defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{20}$$

Now a Riemannian manifold is said to be projectively pseudosymmetric if it satisfies the condition

$$(R(X, Y) \cdot P)(U, V)W = L_P((X \wedge Y) \cdot P)(U, V)W), \tag{21}$$

where $L_P(\neq k)$ is a function on M.

Let us suppose that a $N(K)$-contact metric manifold satisfies the condition

$$(R(X, Y) \cdot P)(U, V)W = L_P((X \wedge Y) \cdot P)(U, V)W). \tag{22}$$

Putting $Y = W = \xi$ (22), we have

$$(R(X, \xi) \cdot P)(U, V)\xi = L_P((X \wedge \xi) \cdot P)(U, V)\xi). \tag{23}$$

Now

$$\begin{aligned} L_P((X \wedge \xi).P)(U, V)\xi) = L_P[&(X \wedge \xi)P(U, V)\xi - P((X \wedge \xi)U, V)\xi \\ &- P(U, (X \wedge \xi)V)\xi - P(U, V)(X \wedge \xi)\xi]. \end{aligned} \tag{24}$$

In view of (1), the projective curvature tensor of a $(2n + 1)$-dimensional $N(k)$-contact manifold is

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y]. \tag{25}$$

Now from the above equation with the help of (14), we have

$$P(U, V)\xi = 0 \tag{26}$$

for any vector fields U, V.

Using (14), (25) and (26) in (24), we get

$$(X \wedge \xi)P(U, V)\xi = 0, \tag{27}$$

$$P((X \wedge \xi)U, V)\xi = 0, \tag{28}$$

$$P(U, (X \wedge \xi)V)\xi = 0, \tag{29}$$

$$P(U, V)(X \wedge \xi)\xi = P(U, V)X. \tag{30}$$

In view of (27), (28), (29) and (30), from (24), we obtain

$$L_P((X \wedge \xi) \cdot P)(U, V)\xi) = -L_P P(U, V)X. \tag{31}$$

Therefore, from (23) and (31) we have

$$(R(X, \xi) \cdot P)(U, V)\xi = -L_P P(U, V)X. \tag{32}$$

It follows that

$$\begin{aligned} R(X, \xi)P(U, V)\xi - P(R(X, \xi)U, V)\xi - P(U, R(X, \xi)V)\xi \\ - P(U, V)R(X, \xi)\xi = -L_P P(U, V)X. \end{aligned} \tag{33}$$

Again using (14), (26) in (33) we have

$$-kP(U, V)X = -L_P P(U, V)X. \tag{34}$$

The above equation yields

$$(L_P - k)P(U,V)X = 0. \tag{35}$$

By assumption $L_P \neq k$, hence we get

$$P(U,V)X = 0, \tag{36}$$

for any vector fields U, V.

Conversely, if $P = 0$, then the equation (22) holds trivially. Thus in view of the above result we can state the following theorem.

Theorem 3.1. *A $(2n+1)$-dimensional $N(k)$-contact metric manifold is projectively pseudosymmetric if and only if it is projectively flat.*

Also projectively flatness implies projectively semisymmetric. Therefore, we can state the following Theorem.

Theorem 3.2. *A $(2n+1)$-dimensional projectively pseudosymetric $N(k)$-contact metric manifold is projectively semisymmetric provided $L_P \neq k$.*

4. Pseudoprojectively flat $N(k)$-contact metric manifolds

An $N(k)$-contact metric manifold is said to be pseudoprojectively flat if

$$g(P(\phi X,Y)Z,\phi W) = 0. \tag{37}$$

From (5), we have

$$\begin{aligned} g(P(\phi X,Y)Z,\phi W) &= g(R(\phi X,Y)Z,\phi W) \\ &\quad - \frac{1}{2n}\{S(Y,Z)g(\phi X,\phi W) - S(\phi X,Z)g(Y,\phi W)\} \end{aligned} \tag{38}$$

for all X, Y, Z and W. Let us take an orthonormal basis $\{e_1, e_2, \ldots, e_{2n}, \xi\}$ in M. Therefore from (38) we get

$$\begin{aligned} \sum_{i=1}^{2n} g(P(\phi e_i,Y)Z,\phi e_i) &= \sum_{i=1}^{2n} g(R(\phi e_i,Y)Z,\phi e_i) \\ &\quad - \frac{1}{2n}\sum_{i=1}^{2n}\{S(Y,Z)g(\phi e_i,\phi e_i) - S(\phi e_i,Z)g(Y,\phi e_i)\}. \end{aligned} \tag{39}$$

In a $(2n+1)$-dimensional almost contact metric manifold, if $\{e_1, e_2, \ldots, e_{2n}, \xi\}$ is a local orthonormal basis of a vector field in M, then $\{\phi e_1, \phi e_2, \ldots, \phi e_{2n}, \xi\}$ is a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i,e_i) = \sum_{i=1}^{2n} g(\phi e_i,\phi e_i) = 2n, \tag{40}$$

$$\begin{aligned} \sum_{i=1}^{2n} g(e_i,Z)S(Y,e_i) &= \sum_{i=1}^{2n} g(\phi e_i,Z)S(Y,\phi e_i) \\ &= S(Y,Z) - S(Y,\xi)\eta(Z) \end{aligned} \tag{41}$$

for all Y, $Z \in TM$.

In an $N(k)$-contact metric manifold, we also have

$$g(R(\xi, Y)Z, \xi) = kg(\phi Y, \phi Z) \tag{42}$$

for all Y, $Z \in TM$. Consequently,

$$\begin{aligned} \sum_{i=1}^{2n} g(R(e_i, Y)X, e_i) &= \sum_{i=1}^{2n} g(R(\phi e_i, Y)Z, \phi e_i) \\ &= S(Y, Z) - kg(\phi Y, \phi Z). \end{aligned} \tag{43}$$

Using the equations (40), (41), (42) and (43) in (39), we have

$$\sum_{i=1}^{2n} g(P(\phi e_i, Y)Z, \phi e_i) = \frac{1}{2n} S(Y, Z) - kg(Y, Z). \tag{44}$$

If the manifold M satisfies (37), we have

$$\frac{1}{2n} S(Y, Z) - kg(Y, Z) = 0. \tag{45}$$

This implies

$$S(Y, Z) = 2nkg(Y, Z). \tag{46}$$

Therefore, a pseudoprojectively flat $N(k)$-contact metric manifold be an Einstein manifold.

Conversely, let the manifold is an Einstein manifold. Then we have

$$S(X, Y) = 2nkg(X, Y). \tag{47}$$

Now

$$\begin{aligned} g(P(\phi X, Y)Z, &\phi W) \\ &= g(R(\phi X, Y)Z, \phi W) - \frac{1}{2n}\{S(Y, Z)g(\phi X, \phi W) \\ &\quad - S(\phi X, Z)g(Y, \phi W)\} \\ &= g(k[g(Y, Z)\phi X - g(\phi X, Z)Y, \phi W) - \frac{1}{2n}\{2nkg(Y, Z)g(\phi X, \phi W) \\ &\quad - 2nkg(\phi X, Z)g(Y, \phi W)\} \qquad \text{using (5.11)} \\ &= k[g(Y, Z)g(\phi X, \phi W) - g(\phi X, Z)g(Y, \phi W) - g(Y, Z)g(\phi X, \phi W) \\ &\quad + (\phi X, Z)g(Y, \phi W)] = 0. \end{aligned} \tag{48}$$

It follows that the manifold is pseudoprojectively flat. Thus in view of the above result, we can state the following theorem.

Theorem 4.1. *A $(2n+1)$-dimensional $N(k)$-contact metric manifold is pseudoprojectively flat if and only if it is an Einstein manifold.*

S. Tanno [**22**] proved the next theorem.

Theorem 4.2 ([**22**])**.** *If M is a $(2n+1)$-dimensional $(n \geq 2)$ Einstein contact metric manifold with ξ belonging to the k-nullity distribution, then $k=1$ and M is Sasakian.*

Therefore we can state the following:

Theorem 4.3. *A $(2n+1)$-dimensional $(n \geq 2)$ pseudoprojectively flat $N(k)$-contact metric manifold is a Sasakian manifold.*

5. $N(k)$-contact metric manifolds satisfying $\widetilde{Z} \cdot P = 0$

In this section, we consider an $N(k)$ contact metric manifold satisfying $\widetilde{Z} \cdot P = 0$. Therefore, we have

$$(\widetilde{Z}(X,Y) \cdot P)(U,V)W = 0. \tag{49}$$

This implies

$$\begin{aligned}\widetilde{Z}(X,Y)P(U,V)W - P(\widetilde{Z}(X,Y)U,V)W - P(U,\widetilde{Z}(X,Y)V)W \\ - P(U,V)\widetilde{Z}(X,Y)W = 0.\end{aligned} \tag{50}$$

Putting $X=\xi$ in (50), we have

$$\begin{aligned}\widetilde{Z}(\xi,Y)P(U,V)W - P(\widetilde{Z}(\xi,Y)U,V)W - P(U,\widetilde{Z}(\xi,Y)V)W \\ - P(U,V)\widetilde{Z}(\xi,Y)W = 0.\end{aligned} \tag{51}$$

Now,

$$\begin{aligned}\widetilde{Z}(\xi,Y)P(U,V)W &= R(\xi,Y)P(U,V)W - \frac{r}{2n(2n+1)}[g(Y,P(U,V)W)\xi \\ &\quad - g(\xi,P(U,V)W)Y], \\ &= k[g(Y,P(U,V)W)\xi - \eta(P(U,V)W)] \\ &\quad - \frac{r}{2n(2n+1)}[g(Y,P(U,V)W)\xi - \eta((U,V)W)Y] \\ &= (k - \frac{r}{2n(2n+1)})[g(Y,P(U,V)W)\xi \\ &\quad - \eta((U,V)W)Y].\end{aligned} \tag{52}$$

Similarly,

$$\begin{aligned}&P(\widetilde{Z}(\xi,Y)U,V)W \\ &\quad = (k - \frac{r}{2n(2n+1)})[g(Y,U)P(\xi,V)W - \eta(U)P(Y,V)W],\end{aligned} \tag{53}$$

$$\begin{aligned}&P(U,\widetilde{Z}(\xi,Y)V)W \\ &\quad = (k - \frac{r}{2n(2n+1)})[g(Y,V)P(U,\xi)W - \eta(V)P(U,Y)W]\end{aligned} \tag{54}$$

and

$$P(U,V)\widetilde{Z}(\xi,Y)W \\ = (k - \frac{r}{2n(2n+1)})[g(Y,W)P(U,V)\xi - \eta(W)P(U,V)Y]. \tag{55}$$

Using (52), (53), (54) and (55) in (51), we obtain

$$\begin{aligned} \Big(k - \frac{r}{2n(2n+1)}\Big)[&g(Y,P(U,V)W)\xi - \eta((U,V)W)Y \\ &- g(Y,U)P(\xi,V)W + \eta(U)P(Y,V)W \\ &- g(Y,V)P(U,\xi)W + \eta(V)P(U,Y)W \\ &- g(Y,W)P(U,V)\xi + \eta(W)P(U,V)Y] = 0. \end{aligned} \tag{56}$$

Therefore, either $k = \frac{r}{2n(2n+1)}$, or

$$\begin{aligned} &g(Y,P(U,V)W)\xi - \eta((U,V)W)Y - g(Y,U)P(\xi,V)W \\ &+ \eta(U)P(Y,V)W - g(Y,V)P(U,\xi)W + \eta(V)P(U,Y)W \\ &- g(Y,W)P(U,V)\xi + \eta(W)P(U,V)Y = 0. \end{aligned} \tag{57}$$

Putting $V = \xi$ in (57), we have

$$\begin{aligned} &g(Y,P(U,\xi)W)\xi - \eta((U,\xi)W)Y - g(Y,U)P(\xi,\xi)W \\ &+ \eta(U)P(Y,\xi)W - g(Y,\xi)P(U,\xi)W + \eta(\xi)P(U,Y)W \\ &- g(Y,W)P(U,\xi)\xi + \eta(W)P(U,\xi)Y = 0. \end{aligned} \tag{58}$$

Using (1) in (58), we have

$$\begin{aligned} &-\Big\{\frac{1}{2n}S(U,W) - kg(U,W)\Big\}Y \\ &+\Big\{\frac{1}{2n}S(Y,W) - kg(Y,W)\Big\}\eta(U)\xi \\ &+ P(U,Y)W + \Big\{\frac{1}{2n}S(U,Y) - kg(U,Y)\Big\}\eta(W)\xi = 0. \end{aligned} \tag{59}$$

Taking inner product of (59) with ξ and using (1), we have

$$\eta(W)[S(U,Y) - 2nkg(U,Y)] = 0 \tag{60}$$

for all vector fields Y, U, W.

Since $\eta(W) \neq 0$, the equation (60) yields $S(U,Y) = 2nkg(U,Y)$. Thus $\widetilde{Z}\cdot P = 0$ implies $k = \frac{r}{2n(2n+1)}$, that is, $r = 2n(2n+1)k$ or Einstein manifold. Again, we know that the scalar curvature of an $N(k)$-contact metric manifold is $r = 2n \cdot (2n-2+k)$. Comparing the values of r, we obtain $k = 1 - \frac{1}{n}$, and hence M is locally isometric to the manifold of Example 2.1 for $n > 1$ and flat for $n = 1$.

Thus in view of the above result, we can state the following.

Theorem 5.1. *If a $(2n+1)$-dimensional non-Sasakian $N(k)$-contact metric manifold satisfies $\widetilde{Z}\cdot P = 0$, then either it is an Einstein manifold or locally isometric to the manifold of Example 2.1 for $n > 1$ and flat for $n = 1$.*

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