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MAXIMUM NUMBER OF TRIANGLE-FREE EDGE COLOURINGS WITH FIVE AND SIX COLOURS

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Abstract. Let $k \geq 3$ and $r \geq 2$ be natural numbers. For a graph G, let $F(G, k, r)$ denote the number of colourings of the edges of G with colours $1, \ldots, r$ such that, for every colour $c \in \{1, ..., r\}$, the edges of colour c contain no complete graph on k vertices K_k. Let $F(n, k, r)$ denote the maximum of $F(G, k, r)$ over all graphs G on n vertices. The problem of determining $F(n, k, r)$ was first proposed by Erdős and Rothschild in 1974, and has so far been solved only for $r = 2, 3$, and a small number of other cases.

In this paper we consider the question for the cases $k = 3$ and $r = 5$ or $r = 6$. We almost exactly determine the value $F(n, 3, 6)$ and approximately determine the value $F(n, 3, 5)$ for large values of n. We also characterise all extremal graphs for $r = 6$ and prove a stability result for $r = 5$.

1. Introduction

A fundamental theorem of graph theory by Turán [**14**] asserts that among the graphs on n vertices that do not contain a complete graph on k vertices K_k, the complete balanced $k-1$-partite graph, also known as the Turán graph $T_{k-1}(n)$, has the largest number of edges $t_{k-1}(n)$. Clearly, no matter how we colour the edges of the Turán graph, the edges of the same colour form a graph with no K_k in it. We call such a colouring K_k-free. Hence $T_{k-1}(n)$ has $r^{t_{k-1}(n)}$ K_k-free colourings with r colours. A natural question is whether we can find a graph with more K_k-free colourings, and if yes, which such graph has the most edges.

Let $k \geq 3$ and $r \geq 2$ be natural numbers. By a colouring of a graph $G = (V, E)$ with r colours we mean a function $f \colon E \to \{1, \ldots, r\}$. In this context we refer to the numbers $1, \ldots, r$ as colours. For a graph G, let $F(G, k, r)$ denote the number of K_k-free colourings of G with r colours. Let $F(n, k, r)$ denote the maximum of $F(G, k, r)$ over all graphs G on n vertices. Then the above lower bound obtained from the Turán graph can be restated as

$$F(n, k, r) \geq r^{t_{k-1}(n)}. \tag{1}$$

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The problem of determining $F(n,k,r)$ was first proposed by Erdős and Rothschild in 1974 [**6, 7**]. They conjectured that in the case $k=3$ and $r=2$ the lower bound (1) is sharp for large enough n, and furthermore that $T_2(n)$ is the unique extremal graph. Their conjecture was proved by Yuster [**15**] who also proved an approximate version of the statement for general k and $r=2$.

Improving on Yuster's results, Alon, Balogh, Keevash and Sudakov fully resolved the $r=2$ and $r=3$ cases for large values of n.

Theorem 1.1 (Alon, Balogh, Keevash, Sudakov [**1**])**.** *For $k \geq 3$ and $n \geq n_0(k)$ the following holds, $F(n,k,2) = 2^{t_{k-1}(n)}$ and $F(n,k,3) = 3^{t_{k-1}(n)}$. Moreover, the corresponding unique extremal graph is $T_{k-1}(n)$.*

In their paper [**1**] the authors also noted that the case $r > 3$ is more challenging as the behavior of $F(n,k,r)$ changes. Indeed, they proved that if $r > 3$ then $F(n,k,r)$ is exponentially larger than $r^{t_{k-1}(n)}$. They also determined the approximate values of $F(n,3,4)$ and $F(n,4,4)$. Subsequently, Pikhurko and Yilma improved on their result showing that $F(n,3,4) = F(T_4(n),3,4)$ and $F(n,4,4) = F(T_9(n),4,4)$ and the corresponding extremal graphs are unique [**11**]. More recently, Pikhurko, Staden and Yilma [**12**] proved that for every n,k,r there is a complete multipartite graph G such that $F(n,k,r) = F(G,k,r)$. This graph is not necessarily unique and not necessarily balanced.

In the current paper we consider the case $k=3$ and $r=5$ or $r=6$. Let $\varphi_r(G) = F(G,3,r)$ and $\varphi_r(n) = F(n,3,r)$. In the case $r=6$ we determine the value of $\varphi_6(n)$ almost exactly. Furthermore, for large values of n, we determine all graphs G for which $\varphi_6(G) = \varphi_6(n)$.

Theorem 1.2. *For every n we have $\varphi_6(n) = (c_n + o(1)) \cdot 3^{n^2/3} 4^{3n^2/16}$, where c_n is a constant depending only on n modulo 8. Moreover, if $\varphi_6(G) = \varphi_6(n)$, then G is a complete balanced 8-partite graph.*

To state our result for the case $r=5$ we need the following notations. Suppose we are given a graph G and subgraphs $H_1,\ldots,H_r \subseteq G$ with $E(G) = E(H_1) \cup \ldots \cup E(H_r)$. We denote by $\Phi(H_1,\ldots,H_r)$ the family of all colourings χ in which $\chi^{-1}(i) \subseteq E(H_i)$ for all $i \in [r]$. If each H_i is triangle-free, then every such colouring is triangle-free, hence we have $\varphi_r(G) \geq |\Phi(H_1,\ldots,H_r)|$. Furthermore, the number of such colourings is easy to count. For $i \in [r]$, let $M_i = M_i(H_1,\ldots,H_r) \subseteq G$ denote the graph on $V(G)$ whose edges are those of G contained in exactly i of the subgraphs $H_1,\ldots,H_r$. We denote by $m_i = m_i(H_1,\ldots,H_r)$ the number of edges in M_i.

Theorem 1.3 (Stability for $r=5$)**.** *For every $\varepsilon > 0$ there is an n_0 such that for all graphs G on $n > n_0$ vertices the following holds. Suppose $H_1,\ldots,H_5 \subseteq G$ and $|\Phi(H_1,\ldots,H_5)| \geq 6^{n^2/4 - \varepsilon n^2/4}$. Then there are balanced bipartitions $V_i^0 \cup V_i^1 = V(H_i)$ for each $i \in [5]$, an integer $t \in \{4,6,8\}$ and $v^{(1)},\ldots,v^{(t)} \in \{0,1\}^5$ such that*

$$\Big| \bigcup_{i\in[t]} \bigcap_{j\in[5]} V_j^{v_j^{(i)}} \Big| \geq \left(1 - 10000\sqrt{\varepsilon}\right) n.$$

Further, there is an $s \in [5]$ and a partition of $\{v^{(1)}, \dots, v^{(t)}\}$ into

$$\mathcal{E}_0 = \left\{v^{(i)} : i \in [t], v_s^{(i)} = 0\right\} \quad \text{and} \quad \mathcal{E}_1 = \left\{v^{(i)} : i \in [t], v_s^{(i)} = 1\right\}$$

such that

1. *each pair $(v, v') \in \mathcal{E}_0 \times \mathcal{E}_1$ has distance three and each pair $(v, v') \in \binom{\mathcal{E}_0}{2} \cup \binom{\mathcal{E}_1}{2}$ has distance two or four.*
2. *Moreover, pairs of distance four form a perfect matching in $\mathcal{E}_0 \times \mathcal{E}_1$ and for each of these matched pairs v, v' we have $|V(v)| = |V(v')| \pm \sqrt[4]{\varepsilon} n$.*

This theorem means that every graph G for which $\varphi_5(G)$ is close to $\varphi_5(n)$, is close to a complete eight-partite graph with one of the following part-sizes: $(n/4, n/4, a, a, b, b, n/4 - a - b, n/4 - a - b)$ for some $0 \le a, b$ and $a + b \le n/4$ or $(a, a, n/4 - a, n/4 - a, b, b, n/4 - b, n/4 - b)$ for some $0 \le a, b \le n/4$. Calculating the number of K_3-free colourings for these graphs gives the following corollary.

Corollary 1.4. *For every n we have $\varphi_5(n) \le 6^{n^2/4 + o(n^2)}$.*

The fact that the aforementioned graphs all have asymptotically equal number of K_3-free colourings makes the case $r = 5$ particularly difficult and interesting. The authors are currently working on finding the exact solution for this case.

2. METHODS

In this section we describe (without proof) the main ingredients of our proof of Theorem 1.2. We also include a brief outline of the proof without technical details. We focus on the case $r = 6$ as the case $r = 5$ uses the same ideas but it is technically more difficult to state.

Our aim is to prove that $T_8(n)$ is the unique extremal graph. Our proof consists of the following three steps.

First, using the container method, we approximate the number of K_3-free colourings of $T_8(n)$ and show that it is approximately optimal.

Second, we prove a structural stability result, that is, any graph that has nearly optimal number of K_3-free colourings must be very similar to $T_8(n)$. For that, we use theorems of Bollobás [**4**] and Füredi [**8**] to investigate the structure of the containers, proving that they must be close to complete balanced bipartite graphs.

Last, we prove the exact result. Starting from the stability result, we prove a series of local improvement claims. By proving a series of technical local improvement claims on our stability result, we strengthen the sense in which the extremal graph is close to $T_8(n)$. After a series of such technical claims we conclude that $T_8(n)$ is indeed the unique extremal graph.

2.1. Approximate upper bounds

The main tool in this part is the container theorem below proved by Mousset, Nenadov and Steger [**10**] using the hypergraph container method of Balogh, Morris and Samotij [**2**] and Saxton and Thomason [**13**]. We use an equivalent formulation of their result as stated by Balogh et al. in [**3**]. This approach was introduced by Hàn and Jiménez [**9**] using ideas of Clemens, Das and Tran [**5**].

Theorem 2.1 ([**3**, Theorem 3.2])**.** *There exists constant n_0 such that for every graph G on $n > n_0$ vertices there exists a collection $\mathcal{C} = \mathcal{C}(G)$ of subgraphs of G such that the following holds:*

(a) *every triangle-free subgraph $G' \subseteq G$ is a subgraph of some $C \in \mathcal{C}$,*
(b) *$K_3(C) \leq n^{25/9}$ for every $C \in \mathcal{C}$,*
(c) *$|\mathcal{C}| \leq \exp\left(n^{16/9}\right)$.*

Although the container theorem leads to an initial upper bound, we need the following stronger result, which is obtained by investigating (as described in the next subsection) the structure of the containers.

Theorem 2.2. *For every graph G on n vertices, the number of 6-colourings of $E(G)$ without monochromatic triangles is at most*

$$3^{n^2/4} 4^{3n^2/16 + o(n^2)}.$$

2.2. Stability

By investigating the structure of the containers, we find that each container either has a low number of edges, which makes it irrelevant for the total number of colourings, or it is very close to complete balanced bipartite. For that, we use the following theorems.

Theorem 2.3 (Bollobás [**4**])**.** *Every graph with n vertices and m edges has at least $\frac{n}{9}(4m - n^2)$ triangles.*

Theorem 2.4 (Füredi [**8**])**.** *Every triangle-free graph with at least $\frac{n^2}{4} - t$ edges has a bipartite subgraph with at least $\frac{n^2}{4} - 2t$ edges.*

The reason for showing that each relevant container is close to bipartite is that then for every six relevant containers $C_1, \dots, C_6$, the graph $M_3(C_1, \dots, C_6)$ (recall that this means the edges that appear in exactly 3 of the containers) does not contain many triangles. This gives us the desired improvement to the approximate upper bound as well as the approximate structure of any graph close to extremal.

Theorem 2.5 (Stability for $r = 6$)**.** *For every $\varepsilon > 0$ there is an n_0 such that for all graphs G on $n > n_0$ vertices the following holds. Suppose $H_1, \dots, H_6 \subseteq G$ and $|\Phi(H_1, \dots, H_6)| \geq 3^{n^2/4} 4^{3n^2/16 - \varepsilon n^2}$. Then there are balanced bipartitions $V_i^0 \cup V_i^1 = V(H_i)$ for each $i \in [6]$, and $v^{(1)}, \dots, v^{(8)} \in \{0,1\}^6$ such that*

$$\Big| \bigcup_{i \in [8]} \bigcap_{j \in [6]} V_j^{v_j^{(i)}}) \Big| \geq \left(1 - 10000\sqrt{\varepsilon}\right) n.$$

Further, there is a partition of $\{v^{(1)}, \dots, v^{(8)}\}$ into $\mathcal{E}_0$ and $\mathcal{E}_1$ with $|\mathcal{E}_0| = |\mathcal{E}_1|$ such that each pair $(v, v') \in \mathcal{E}_0 \times \mathcal{E}_1$ has distance three and each pair $(v, v') \in \binom{\mathcal{E}_0}{2} \cup \binom{\mathcal{E}_1}{2}$ has distance four.

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