Acta Math. Univ. Comenianae
Vol. LXXXVIII, 3 (2019), pp. 1093–1100

LENGTH OF CYCLES IN GENERALIZED PETERSEN GRAPHS

Z.-B. ZHANG AND Z. CHEN

ABSTRACT. There have been extensive researchs on cycles in regular graphs, particularly 3-connected cubic graphs. Generalized Petersen graphs, denoted by $GP(n,k)$, are highly symmetric 3-connected cubic graphs, which have attracted great attention. The Hamiltonicity of $GP(n,k)$ has been studied for a long time and thoroughly settled. Inspired by Bondy's meta-conjecture that almost every nontrivial condition for Hamiltonicity also implies pancyclicity, we seek for more cycle structures in this class of graphs, by figuring out the possible lengths of cycles in them.

It turns out that generalized Petersen graphs, though not generally pancyclic, miss only very few possible length of cycles. For $k \in \{2,3\}$, we completely determine all possible cycle lengths in $GP(n,k)$. We also obtain some results for $GP(n,k)$ where k is odd. In particular, when k is odd, and n is even and sufficiently large, $GP(n,k)$ is bipartite and weakly even pancyclic.

1. INTRODUCTION

Theory on cycles in graphs forms an important branch of graph theory. *Hamiltonian cycles*, i.e. cycles that go through every vertex of a graph exactly once, have caught particularly attention. It can be observed that in many occasions the existence of Hamiltonian cycles also accompany with cycles of many or even all possible lengths. In 1970's, Bondy introduced the concept of *pancyclic graphs*, which are graphs that contain cycles of every length k for $3 \le k \le |G|$, and posed the meta-conjecture that almost every nontrivial condition for hamiltonicity also implies pancyclicity. As an illustration of this idea, he showed that Ore's classical condition which states that $d(u) + d(v) \ge |G|$ for any nonadjacent vertex pair $\{u,v\} \in V(G)$, implies not only hamiltonicity but also pancyclicity, while excluding complete balanced bipartite graphs. From then on, numerous research works came forth to generalize various Hamiltonian conditions to pancyclicity.

It is usually easier to find rich cycle structures in dense graphs. For example, Ore's condition implies that the graph G contains at least $(|G|^2 + 1)/4$ edges, and thus is quite dense. On the other hand, it is relatively challenging to find Hamiltonian cycles or cycles of specified lengths in sparse graphs. One typical

Received May 21, 2019.
2010 *Mathematics Subject Classification.* Primary 05C38.
The first author was supported by Natural Science Foundation of Guangdong Province (2016A030313829), the Key Research Project of Universities in Guangdong Province (2018GKZDXM004), and the Talent Project of Guangdong Industry Polytechnic (RC2016-004 and 2B141403).

kind of sparse graphs that are of broad interest is k-regular graphs with a fixed degree k. Jackson ([**6**]) proved that a 2-connected k-regular graph with at most $3k$ vertices is Hamiltonian. Zhu et al. ([**12**]) improved the result to include graphs with $3k+1$ vertices. It is well-known that Tait had conjectured ([**10**]) every 3-connected cubic planar graph contains a Hamilton cycle, but this conjecture was disproved by a counterexample found by Tutte ([**11**]).

In general k-regular graphs with larger order with respect to k, it is less likely that a Hamiltonian cycle exists. However, there are continuous works to find out the length of the longest cycle, i.e., the *circumference* of such graphs. The circumference of a graph G is denoted by $c(G)$. Improving several former results, Liu et al. ([**7**]) recently proved a lower bound of $\Omega(n^{0.8})$ for the circumference of 3-connected cubic graphs. The best upper bound so far was proved by Bondy and Simonovits ([**5**]) who constructed an infinite family of 3-connected cubic graphs with circumference $\Theta(n^{\log_9 8}) \approx \Theta(n^{0.946})$.

In this paper, we consider cycles in a highly symmetric subclass of 3-connected cubic graphs, namely the generalized Petersen graphs. A *generalized Petersen graph*, denoted by $GP(n,k)$, consists of the vertex set $U = \{u_i : 0 \le i \le n-1\} \cup V = \{v_i : 0 \le i \le n-1\}$ and the edge set $\{u_iu_{i+1}, v_iv_{i+k}, u_iv_i : 0 \le i \le n-1\}$, where $n \ge 5$, $k < n/2$, and the subscripts of the vertices are modulo n. Generalized Petersen graphs were introduced by Watkins when studying edge-coloring of cubic graphs. The name comes from the *Petersen graph*, which is exactly $GP(5,2)$. After their introduction, generalized Petersen graphs have been studied from many aspects such as transitivity, hamiltonicity and coloring.

The study of Hamiltonian cycles in generalized Petersen graphs can be traced back to the doctoral thesis ([**9**]) of Robertson. Through continuous works of Bondy ([**4**]), Bannai ([**3**]) and Alspach ([**1**]), the problem was thoroughly solved, as summarized in the theorem below.

Theorem 1.1. *$GP(n,k)$ is Hamiltonian, unless $GP(n,k) \cong GP(6t+5,2)$ for a certain $t \ge 0$. Furthermore every $GP(n,k) \cong GP(6t+5,2)$ contains a cycle of length $2n-1$.*

A graph is *Hamiltonian-connected* if every vertex pair $\{u,v\}$ is joined by a Hamiltonian path. In some recent works ([**2, 8**]), the Hamiltonian-connectedness of $GP(n,k)$ is studied.

Following the idea of the meta-conjecture of Bondy, we will focus on pancyclicity and related problems of generalized Petersen graphs in this paper. Firstly we introduce some variants of pancyclicity. The length of the shortest cycle in a graph G, denoted by $g(G)$, is called the *girth* of G. In the case that G contains cycles of every length between $g(G)$ and $c(G)$, we say that G is *weakly pancyclic.* When G is bipartite, G contains cycles of even lengths only. Correspondingly, we define *even pancyclicity* and *weak even pancyclicity.* Note that even the Petersen graph itself, which contains cycles of length $5,6,8$ and 9, is not pancyclic. As it turns out, in many cases, $GP(n,k)$ is not pancyclic or weakly pancyclic, but only misses very few length of cycles. Therefore, the main theme of our work is figuring out the set of lengths of cycles that we can find in $GP(n,k)$.

To find the possible cycle lengths in $GP(n, k)$, it is helpful to firstly make clear their accurate bounds, i.e., the circumference and the girth of $GP(n, k)$. The circumference of $GP(n, k)$ is stated in Theorem 1.1. Our first main result completely determines the girth of $GP(n, k)$.

Theorem 1.2.

$$g(GP(n,k)) = \min\Big(8, \frac{n}{\gcd(n,k)}, k+3, n-(k-1)\lfloor\frac{n}{k}\rfloor+2, (1+k)\lceil\frac{n}{k}\rceil - n + 2\Big).$$

Then, we determine all possible cycle lengths of $GP(n, k)$ for $k \in \{2, 3\}$, in the two theorems below.

Theorem 1.3. *The possible lengths of cycles in $GP(n, 2)$ are as follows.*

(1) *All $GP(n, 2)$ contain l-cycles for $l = 5$ and $8 \le l \le 2n - 2$;*
(2) *A $GP(n, 2)$ contains $2n$-cycles (i.e., Hamiltonian cycles) iff $n \not\equiv 5 \pmod 6$;*
(3) *A $GP(n, 2)$ contains $(2n - 1)$-cycles iff $n \not\equiv 4 \pmod 6$;*
(4) *A $GP(n, 2)$ contains 7-cycles iff $n \in \{6, 7, 8, 9, 14\}$;*
(5) *A $GP(n, 2)$ contains 6-cycles iff $n \in \{5, 6, 7, 12\}$;*
(6) *A $GP(n, 2)$ contains 4-cycles iff $n = 8$;*
(7) *A $GP(n, 2)$ contains 3-cycles iff $n = 6$.*

Theorem 1.4. *The possible lengths of cycles in $GP(n, 3)$ are as follows.*

(1) *When n is even, $GP(n, k)$ is bipartite, and is weakly even pancyclic;*
(2) *When n is odd, $GP(n, k)$ contains cycles of every even length between its girth and circumference, cycles of every odd length between $\lfloor\frac{n-1}{k}\rfloor + 3$ and $2n - 1$, and cycles of length $\frac{n}{k}$ if n is a multiple of k.*

For the general cases, we obtain some results on cycles in $GP(n, k)$ where k is odd, summarized in the theorem below, of which Theorem 1.4 is actually a special case. These results indicate that $GP(n, k)$ contains a wide range of cycle lengths. In particular, when n is even and sufficiently large, $GP(n, k)$ is bipartite and weakly even pancyclic.

Theorem 1.5. *When k is odd, we can find cycles of the following lengths in $GP(n, k)$.*

(1) *$GP(n, k)$ contains cycles of every even length between $4k$ and $2n + 2 - k$.*
(2) *If $n \ge k^2 + \frac{7}{2}k - 4$, then $GP(n, k)$ contains cycles of every even length between $g(GP(n, k))$ to $c(GP(n, k)) = 2n$. Particularly, if n is even, then $GP(n, k)$ is bipartite and is weakly even pancyclic.*
(3) *Only when n is odd, $GP(n, k)$ contains cycles of odd lengths. Let n' be the remainder when n is divided by k, and let $h(n')$ be defined as below:*

$$h(n') = \begin{cases} 6, & \text{if } n' = 0, \\ 3, & \text{if } n' = 1, \\ 6 + n', & \text{if } 2 \le n' < \frac{k+1}{2}, \\ k + 7 - n', & \text{if } n' \ge \frac{k+1}{2}. \end{cases} \tag{1}$$

$GP(n, k)$ contains cycle of every odd length between $\lfloor\frac{n}{k}\rfloor + h(n')$ and $2n + 2 - k$. Furthermore if $n' > 1$, then $GP(n, k)$ contains cycle of length $\lfloor\frac{n}{k}\rfloor + h(n') - 4$, and if $n' = 0$, then $GP(n, k)$ contains cycle of length $\lfloor\frac{n}{k}\rfloor$.

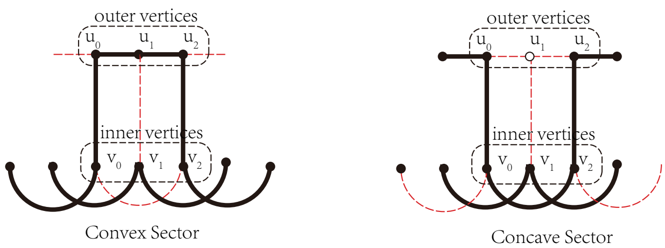

Figure 1. A convex sector and a concave sector.

We conclude the section with some definitions and notations. In a $GP(n,k)$, the vertices in U are called the *outer vertices*, and the cycle they span is called the *outer cycle*. The vertices in V are called the *inner vertices*. Inner vertices may span one cycle or a set of independent cycles, called the *inner cycles*. Inner and outer cycles are both called *trivial cycles*, and all other cycles are called *nontrivial cycles*. Furthermore, we call an edge on the outer cycle an *outer edge*, an edge on an inner cycle an *inner edge*, and an edge between U and V a *spoke*. A cycle of length l is called an l-cycle. We use $\gcd(a,b)$ to denote the greatest common divisor of a and b.

2. Our tools and methods

2.1. Sector

The main tool we use is sector, a concept that we newly introduce. Since trivial cycles are only of length either n or $\frac{n}{\gcd(n,k)}$, we rely on nontrivial cycles for more lengths. While a nontrivial cycle contains at least two spokes, We cut it along spokes into small parts called sectors, defined as below.

Let $0 \leq s,t \leq n-1$, the subgraph of $GP(n,k)$ consisting of the edge sets $\{u_iu_{i+1} : s-1 \leq i \leq t\}$, $\{v_jv_{j+k} : s-k \leq j \leq t\}$ and $\{u_sv_s, u_tv_t\}$ is called a *base* (of sectors) between two spokes u_sv_s and u_tv_t. A *sector* of $GP(n,k)$ is defined to be a subgraph of a base containing u_sv_s and u_tv_t, and in which every u_i and every v_i, $s \leq i \leq t$, is of degree 0 or 2. It can be deduced from the definition that a sector does not contain any of the spokes u_jv_j, $t < j < s$. Furthermore, in a sector every u_i, $t < i < s$, must be of degree 2 or 0 at the same time. In the former case the path $u_su_{s+1}\ldots u_{t-1}u_t$ is contained in the sector, we say that the sector is *convex*; In the latter case no internal vertex of the path $u_su_{s+1}\ldots u_{t-1}u_t$ is in the sector, we say that the sector is *concave*. See Figure 1 for examples of convex and concave sectors.

Let S be a sector between u_sv_s and u_tv_t. The *length* of the sector is defined to be $L(S) = t-s \pmod{n}$. Let

$$\varphi(s) = \begin{cases} 1, & \text{if } v_sv_{s+k} \in E(S) \\ 0, & \text{otherwise.} \end{cases}$$

The *input* of the sector is defined to be

$$I(S) = (\varphi(s-k), \varphi(s-k+1), \ldots, \varphi(s-1)).$$

And the *output* of the sector is defined to be

$$O(S) = (\varphi(t+1-k), \varphi(t+2-k), \ldots, \varphi(t)).$$

The *deficiency* of the sector, denoted by $Q(S)$, is defined to be the number of vertices in $U \cup V$ but not in the sector.

Let $x = (x_0, x_1, \ldots, x_{k-1})$ be a 0-1 k-vector. We define three kinds of functions on x, which are the identity function

$$e(x) = x,$$

the rotating left function

$$r(x) = (x_1, x_2, \ldots, x_{k-1}, x_0),$$

and the inverse function on the i-th component, for $0 \le i \le k-1$,

$$n_i(x) = (x_0, x_1, \ldots, 1 - x_i, \ldots, x_{k-1}).$$

For general integer i, we further define $n_i(x) = n_{i \bmod k}(x)$. Note that r^{-1} is well-defined and exactly the rotating right function. We have $r^k = r^{-k} = n_i^2 = e$, $n_i n_j = n_j n_i$ and $n_i r = r n_{i+1}$.

A sector S is determined by $I(S)$, $L(S)$ and its concavity. In particular, the output of S can be computed as below.

Lemma 2.1. *Let S be a sector of $GP(n,k)$, then*

$$O(S) = r^{L(S)+1} n_{L(S)} n_0(I(S)) = n_{-1} r^{L(S)+1} n_0(I(S)).$$

We can form a cycle by joining sectors. Any two consecutive sectors, say S_1 and S_2, on a nontrivial cycle must be of different concavity. Further, their input and output must satisfy the following equation:

$$r n_0(I(S_2)) = O(S_1), \tag{2}$$

in which case we say that the ordered pair (S_1, S_2) is a *compatible pair*, where S_1 is called the *predecessor* and S_2 is called the *successor*.

Also, the deficiency of a sector can be calculated from its input, length and concavity.

Lemma 2.2. *Let S be a sector in $GP(n,k)$, and let*

$$(o_0, o_1, \ldots, o_{m-1}) = r n_0(I(S_2)),$$

i.e., the output of a predecessor of S on any nontrivial cycle. If S is convex, then $Q(S) = \sum_{j=0}^{L(S)-2} \overline{o_j}$, and if S is concave, then $Q(S) = L(S) - 1 + \sum_{j=0}^{L(S)-2} \overline{o_j}$, where $\overline{o_i} = 1 - o_i$ and $o_l = o_{l-m}$ for $l \ge m$.

Thus, if we form a cycle C with m sectors, say $\{S_0, S_1, \ldots, S_{m-1}\}$, where S_j and S_{j+1} are consecutive on C, then m must be even, S_j must be convex and concave alternatively, and the output of S_j and the input of S_{j+1} must satisfy (2)

($0 \leq j \leq m-1$ and addition modulo m). We define the *deficiency* of a cycle C to be $Q(C) = 2n - |C|$, then $Q(C) = \sum_{i=1}^{m} Q(S_i)$.

Our main method to construct a cycle of a specified length in $GP(n,k)$ is by joining an appropriate set of sectors $\{S_0, S_1, \dots, S_{m-1}\}$ as mentioned above. However, the graph obtained this way may be a disjoint union of more than one cycles. To get over this problem, we introduce a second method to construct cycles. We start with a cycle C'. By replacing some sectors in C' with longer sectors while maintaining the connectedness, we obtain a longer cycle C. We regard such an operation as expansion of a sector, formally defined in the next section.

2.2. Expansion of sectors and primitive cycles

The *p-expansion* of a sector S is an operation which increase the length of S by p, with the input and concavity of S remain the same. If the output of S does not change after a p-expansion, we say that S is *p-expansible*. If $p < L(S)$, a *p-contraction* is an operation which reduce the length of S by p, with the input and concavity of S remain the same. If the output of S does not change after a p-contraction, we say that S is *p-contractible*. In this work, we consider p-expansion and p-contraction for $p \in \{1, k\}$ in $GP(n,k)$.

Firstly, we show that k-expansions can be generally applied.

Lemma 2.3. *In $GP(n,k)$, every sector is k-expansible, and every sector of length at least $k+1$ is k-contractible.*

And, a k-expansion on a sector always transform a cycle into a cycle.

Lemma 2.4. *Let C be a nontrivial cycle in $GP(n,k)$ and S a sector of C. Let S' be the k-expansion of S, and C' be the subgraph of $GP(n+k,k)$ obtained by replacing S with S' in C. Then C' is a cycle.*

For a specified kind of sectors, we have results similar to Lemma 2.3 and 2.4 for 1-expansion.

Lemma 2.5. *Let S be a sector in $GP(n,k)$ and $I(S) = (1, 0, \dots, 0)$. Then S is 1-expansible, and if the length of S is at least 2 then it is 1-contractible.*

Lemma 2.6. *Let C be a nontrivial cycle in $GP(n,k)$ and S a sector of C with $I(S) = (1, 0, \dots, 0)$. Let S' be the 1-expansion of S, and C' be the subgraph of $GP(n+1,k)$ obtained by replacing S with S' in C. Then C' is a cycle.*

Hence, we have a new way to get cycles we desire by expanding sectors on shorter cycles. But we need to make clear the change on deficiency after an expansion. Let S' be a sector obtained from S by a p-expansion. By Lemma 2.2, and following its notations as well, if S and S' are convex, then the deficiency of the sector is increased by

$$\sum_{j=0}^{L(S)+k-2} \overline{o_j} - \sum_{j=0}^{L(S)-2} \overline{o_j} = \sum_{j=0}^{k} \overline{o_j}, \tag{3}$$

and if S and S' are concave, then the deficiency of the sector is increased by

$$(L(S)+k)-1+\sum_{j=0}^{L(S)+k-2}\overline{o_j}-(L(S)-1+\sum_{j=0}^{L(S)-2}\overline{o_j})=k+\sum_{j=0}^{k}\overline{o_j}. \tag{4}$$

Lemma 2.3 and Lemma 2.5 describe two kinds of contractible sectors. Any cycle containing such kinds of sectors can always be contracted. Thus, we choose to start our construction from cycles that do not contain these sectors, which we call *primitive cycles*.

The problem that follows is how to construct primitive cycles. We notice that when constructing cycles, some sectors can continuously repeat any number of times, and by repeating of sectors, we get a class of cycles of similar pattern. To classify and construct primitive cycles, we introduce a representation of them called connecting diagram. A *connecting diagram* for a class of cycles in $GP(n,k)$ is a digraph in which vertices stands for convex sectors and arcs stands for concave sectors. As shown in Figure 2, a convex sector is denoted by a box. The length of the sector is marked down inside the box, while the input and the output of the sector are written down on two sides of the box, respectively. A concave sector is denoted by an arc joining two boxes (which can be the same), and the length of the sector is written down above the arc, or ignored if it is exactly one. Since the input and output of a concave sector is determined by its predecessor and successor, they are not explicitly shown in the figure. Figure 2 exhibits a connecting diagram and the class of cycles in $GP(n,2)$ it represents, in which the sector B can repeat any number of times. P is a concave sector of length 1 and input $(0,1)$. Z is a concave sector of input $(1,0)$ and can be of any positive length. It is a "gap" that the cycle does not covered. Formally speaking, the cycle we form contains none of its vertices except the endvertices of its spokes. Any closed walk in the connecting diagram will then represent a primitive cycle. But any cycle can contain at most one copy of Z, hence it is drawn as a dashed line in Figure 2, and can only be visited once.

To find an l-cycle in $GP(n,k)$, we consider its deficiency $Q=2n-l$. We can expand a primitive cycle with deficiency $Q_0<Q$ in $GP(n_0,k)$ where $n_0<n$ such that the expansions increase the outer cycle by $n-n_0$ and the deficiency by $Q-Q_0$. These requirements forms two equations we called the *expansion equations*. The existence of an l-cycle in $GP(n,k)$ is then depend on whether we could find the appropriate primitive cycle, the corresponding expansion operations, so that the expansion equations have integral solution.

3. Conclusion

We introduce the concept of sector, expansion and primitive cycles, as well as the tool of connecting diagram, which are helpful to construct cycles of specify length in generalized Petersen graphs. Applying these ideas and tools, we find out all possible lengths of cycles in $GP(n,k)$ for $k\in\{2,3\}$. For all odd number k, we obtain a wide range of lengths of cycles, and show that when n is even and sufficiently large, $GP(n,k)$ is weakly even pancyclic.

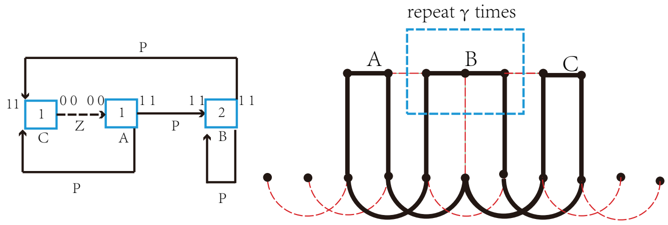

Figure 2. A connected diagram and the class of cycles it represents.

References

1. Alspach B., *The classification of hamiltonian generalized Petersen graphs*, J. Combin. Theory Ser. B **34** (1983), 293–312.
2. Alspach B. and Liu J., *On the Hamilton connectivity of generalized Petersen graphs*, Discrete Math. **309** (2009), 5461–5473.
3. Bannai K., *Hamiltonian cycles in generalized Petersen graphs*, J. Combin. Theory Ser. B **24** (1978), 181–188.
4. Bondy J. A., *Variations on the hamiltonian theme*, Canad. Math. Bull. **15** (1972), 57–62.
5. Bondy J. A. and Simonovits M., *Longest cycles in* 3*-connected* 3*-regular graphs*, Canad. J. Math. **32** (1980), 987–992.
6. Jackson B., *Hamilton cycles in regular* 2*-connected graphs*, J. Combin. Theory **29** (1980), 27–46.
7. Liu Q., Yu X. and Zhang Z., *Circumference of* 3*-connected cubic graphs*, J. Combin. Theory Ser. B **128** (2018), 134–159.
8. Richter R. B., *Hamilton paths in generalized Petersen graphs*, Discrete Math. **313** (2013), 1338–1341.
9. Robertson N., *Graphs Minimal under Girth, Valency, and Connectivity Constraints*, Phd thesis, University of Waterloo, 1969.
10. Tait P. G., *Remarks on the colouring of maps*, Proc. Roy. Soc. Edinburgh **10** (1880), 501–503.
11. Tutte W. T., *On hamiltonian circuits*, J. Lond. Math. Soc. **1** (1946), 98–101.
12. Zhu Y., Liu Z. and Yu, Z., *An improvement of Jackson's result on Hamilton cycles in* 2*-connected regular graphs*, North-holland Mathematics Studies 115, 1985, 237–247.

Z.-B. Zhang, School of Computer Engineering, University of Electronic Science and Technology of China, Zhongshan Institute, Zhongshan, China;
School of Information Technology, Guangdong Industry Polytechnic, Guangzhou, China,
e-mail: `eltonzhang2001@gmail.com`

Z. Chen, International Department, The Affiliated High School of South China Normal University, Guangzhou, China,
e-mail: `emmachen2002@outlook.com`