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EDGE-COLORING OF PLANE GRAPHS WITH MANY COLORS ON FACES

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Abstract. For a fixed positive integer p, a coloring of the edges of a multigraph G is called p-acyclic coloring if every cycle C in G contains at least $\min\{|C|, p+1\}$ colors. The least number of colors needed for a p-acyclic coloring of G is the p-arboricity of G. This type of coloring was introduced by Nešetřil, Ossona de Mendez, and Zhu in 2014. From a result of Bartnicki et al. (2019) it follows that there are planar graphs with unbounded p-arboricity. In this note we improve a result of Bartnicki et al. on p-arboricity of planar graphs with large girth. In addition, we relax the definition of p-arboricity for plane multigraphs in sense that the requirement is not for all cycles but only for facial ones, and we show that the smallest number of colors needed for such a coloring is a constant (depending on p only).

1. Introduction

All graphs considered in this paper are loopless, parallel edges are allowed provided that it is not stated otherwise. We use standard graph theory terminology according to [**3**]. However, the most frequent notions of the paper are defined through it. A plane graph is a particular drawing of a planar graph in the Euclidean plane such that no edges intersect. Let G be a connected plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. The boundary of a face f is the boundary in the usual topological sense. It is the collection of all edges and vertices contained in the closure of f that can be organized into a closed walk in G traversing along a simple closed curve lying just inside the face f. This closed walk is unique up to the choice of initial vertex and direction, and is called the boundary walk of the face f (see [**11**, p. 101]).

Two vertices (two edges) are adjacent if they are connected by an edge (have a common endvertex). A vertex and an edge are incident if the vertex is an endvertex of the edge. A vertex (or an edge) and a face are incident if the vertex (or the edge) lies on the boundary of the face. Two edges of a plane graph G are facially adjacent if they are consecutive on the boundary walk of a face of G.

An edge-coloring of a graph G is an assignment of colors to the edges, one color to each edge. An edge-coloring c of a graph G is proper if for any two adjacent edges e_1 and e_2 of G, $c(e_1) \neq c(e_2)$ holds.

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A facial edge-coloring c of a plane graph G is an edge-coloring such that for any two facially adjacent edges e_1 and e_2 of G, $c(e_1) \neq c(e_2)$ holds. Observe that this coloring need not to be proper in a usual sense. We require only that facially adjacent edges must receive different colors. There are many results concerning different types of facial edge-colorings of plane graphs, see e.g. recent surveys [**4**] and [**5**].

A proper edge-coloring of a simple graph G is r-acyclic if every cycle C contained in G is colored with at least $\min\{|C|, r\}$ colors. The r-acyclic chromatic index $a'_r(G)$ of G is the minimum number of colors needed for an r-acyclic edge-coloring. Clearly, at least Δ colors are required for any r-acyclic edge-coloring of a graph with maximum degree Δ. A 2-acyclic edge-coloring coincides with a proper edge-coloring, thus by the well-known Vizing's theorem [**14**], $\Delta(G) \leq a'_2(G) \leq \Delta + 1$ for every (simple) graph G. Alon et al. [**1**] showed that the 3-acyclic chromatic index is linear in Δ. So far, the best upper bound for $a'_3(G)$ is $\lceil 3.74(\Delta - 1)\rceil + 1$, which was obtained by Giotis et al. [**9**]. 3-acyclic edge-coloring has been deeply studied for planar graphs. The best known upper bound for this class of graphs is $\Delta + 6$, see [**15**].

For $r \geq 4$ the situation is different, in this case the r-acyclic chromatic index may not be linear. In general, for every $r \geq 4$ there exist graphs with $a'_r(G) \geq c_r \cdot \Delta^{\lfloor \frac{r}{2} \rfloor}$, where c_r is a constant (it depends only on r), see [**10**]. On the other hand, if the girth of G is sufficiently large, then the r-acyclic chromatic index is linear in Δ if r is fixed. Gerke and Raemy [**8**] proved that for every graph G with girth at least $3(r-1)\Delta$ it holds that $a'_r(G) \leq 6(r-1)\Delta$. In their theorem the girth is related to both r and Δ. Ding, Wang, and Wu [**6**] also proved a linear upper bound in terms of girth and maximum degree, but in their result the girth depends only on r. They proved that every graph with girth at least $2r - 1$ admits an r-acyclic edge-coloring with at most $(9r-7)\Delta + 10r - 12$ colors. There is only one known result on r-acyclic edge-coloring of planar graphs with $r \geq 4$. Zhang et al. [**16**] showed that $a'_4(G) \leq 37\Delta(G)$ for every planar graph G.

The improper version of this problem is known as a generalized arboricity in the literature. The arboricity of a graph G is the minimum number of colors needed to color the edges of G so that every color class induces a forest. Therefore, if we require that every cycle gets at least two colors, then the minimum number of colors needed is the arboricity of G, and its determination is solved by the well-known Nash-Williams' formula [**12**]. This concept was generalized by Nešetřil, Ossona de Mendez, and Zhu [**13**] in 2014. They introduced the p-arboricity of a graph G as the minimum number of colors needed to color the edges of G in such a way that every cycle C gets at least $\min\{|C|, p+1\}$ colors. Obviously, the 1-arboricity is the classical arboricity.

We have relaxed the definition of p-arboricity for plane graphs as follows:

For a fixed positive integer p, a facial edge-coloring of a plane graph G is called facial p-coloring with respect to faces, if there are at least $\min\{|E(f)|, p\}$ colors on the boundary of each face f, where $E(f)$ denotes the set of edges incident with the face f. The facial p-arboricity of a plane graph G with respect to faces, denoted by $\mathrm{arb}_{\mathrm{rf}}(p, G)$, is the smallest number of colors needed for a facial p-coloring with respect to faces.

Bartnicki et al. [**2**] showed that the p-arboricity cannot be generally bounded by any constant (depending on p only) even for planar graphs, for all $p \geq 2$. Our main result is that the facial p-arboricity of a connected plane graph G can be bounded by a constant depending on p only, which is a significant difference between the facial p-arboricity and the p-arboricity.

2. p-arboricity of planar graphs with large girth

Bartnicki et al. [**2**] proved that there are planar graphs with unbounded p-arboricity, for all $p \geq 2$. On the other hand, they proved that planar graphs with large girth have p-arboricity equal to $p + 1$.

Theorem 1 ([**2**]). *The p-arboricity of every planar graph G of girth at least 2^{p+1} is $p + 1$, for all $p \geq 2$.*

The following statement is an improvement of this result. In our theorem, the girth is bounded by a linear function of p.

Theorem 2. *The p-arboricity of every planar graph G of girth at least $5p + 1$ is $p + 1$, for all $p \geq 2$.*

3. Facial p-arboricity of plane graphs with respect to faces

It is known that every connected plane graph has a facial edge-coloring with at most 4 colors, moreover this bound is tight, see e.g. [**7**]. From the fact that in every facial edge-coloring every face has at least two colors on the boundary it follows that $\mathrm{arb}_{\mathrm{rf}}(2, G) \leq 4$ for every connected plane graph G.

Theorem 3. *If $p \geq 3$ is an integer and G a connected plane graph, then $\mathrm{arb}_{\mathrm{rf}}(p, G) \leq \frac{3}{2}p$. Moreover, this bound is tight.*

There are 2-edge-connected plane graphs G with $\mathrm{arb}_{\mathrm{rf}}(p, G) = \frac{3}{2}p$. For 3-edge-connected plane graphs the bound $\frac{3}{2}p$ can be significantly improved.

Theorem 4. *If $p \geq 3$ is an integer and G a 3-edge-connected plane graph, then*

$$\mathrm{arb}_{\mathrm{rf}}(p, G) \leq \begin{cases} p + 1 & \text{if } p = 3, \text{ or } p \geq 11, \\ p + 2 & \text{if } 4 \leq p \leq 10. \end{cases}$$

Note that for every $p \geq 4$ there are 3-edge-connected plane graphs G with $\mathrm{arb}_{\mathrm{rf}}(p, G) = p$ and 3-edge-connected plane graphs H with $\mathrm{arb}_{\mathrm{rf}}(3, H) = 4$.

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