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ORDERED GRAPHS AND LARGE BI-CLIQUES IN INTERSECTION GRAPHS OF CURVES

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Abstract. An *ordered graph* $G_<$ is a graph with a total ordering $<$ on its vertex set. A *monotone path* of length k is a sequence of vertices $v_1 < v_2 < \cdots < v_k$ such that v_iv_j is an edge of $G_<$ if and only if $|j-i| = 1$. A *bi-clique* of size m is a complete bipartite graph whose vertex classes are of size m.

We prove that for every positive integer k, there exists a constant $c_k > 0$ such that every ordered graph on n vertices that does not contain a monotone path of length k as an induced subgraph has a vertex of degree at least c_kn, or its complement has a bi-clique of size at least $c_kn/\log n$. A similar result holds for ordered graphs containing no induced ordered subgraph isomorphic to a fixed ordered matching.

As a consequence, we give a short combinatorial proof of the following theorem of Fox and Pach. There exists a constant $c > 0$ such the intersection graph G of any collection of n x-monotone curves in the plane has a bi-clique of size at least $cn/\log n$ or its complement contains a bi-clique of size at least cn. (A curve is called x-monotone if every vertical line intersects it in at most one point.) We also prove that if G has at most $\left(\frac{1}{4}-\epsilon\right)\binom{n}{2}$ edges for some $\epsilon > 0$, then $\overline{G}$ contains a linear sized bi-clique. We show that this statement does not remain true if we replace $\frac{1}{4}$ by any larger constants.

1. Ordered graphs

There are a growing number of examples showing that ordered structures can be useful for solving geometric and topological problems that appear to be hard to analyze by traditional combinatorial methods. The aim of the present note is to provide an example concerning intersection patterns of curves, where one can apply ordered graphs.

An *ordered graph* $G_<$ is a graph G with a total ordering $<$ on its vertex set. If the ordering $<$ is clear from the context, we write G instead of $G_<$. An ordered graph $H_{<'}$ is an *induced subgraph* of the ordered graph $G_<$, if there exists an embedding $\phi\colon V(H) \to V(G)$ such that for every $u, v \in V(H)$, if $u <' v$ then $\phi(u) < \phi(v)$, and $uv \in E(H)$ if and only if $\phi(u)\phi(v) \in E(G)$.

A *monotone path* P_k of length k is an ordered graph with k vertices $v_1 < v_2 < \cdots < v_k$ in which v_iv_j is an edge if and only if $|j-i| = 1$. A *bi-clique* in an (ordered or unordered) graph G consists of a pair of disjoint subsets of the vertices (A, B)

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such that $|A| = |B|$ and for every $a \in A$ and $b \in B$, there is an edge between a and b. The size of a bi-clique (A, B) is $|A|$. A *comparability graph* is a graph G for which there exists a partial ordering on $V(G)$ such that two vertices are joined by an edge of G if and only if they are comparable by this partial ordering. An *incomparability graph* is the complement of a comparability graph. The maximum degree of the vertices of G is denoted by $\Delta(G)$.

Our first theorem states that if a P_k-free ordered graph is not too dense, then its complement contains a large bi-clique.

Theorem 1. *For every integer $k \geq 2$, there exists a constant $c = c(k) > 0$ such that the following statement is true. Let $G_<$ be an ordered graph on n vertices which satisfies $\Delta(G_<) < cn$ and does not have any induced ordered subgraph isomorphic to the monotone path P_k of length k.*

Then the complement of $G_<$ contains a bi-clique of size at least $cn/\log n$.

For the conclusion to hold, we need some upper bound on the degrees of the vertices (or on the number of edges) of the graph. To see this, consider the graph G on the naturally ordered vertex set $\{1, \ldots, n\}$, in which $A = \{1, \ldots, \lfloor n/2 \rfloor\}$ and $B = \{\lfloor n/2 \rfloor + 1, \ldots, n\}$ induce complete subgraphs, and any pair of vertices $a \in A, b \in B$ are joined by an edge randomly, independently with a very small probability $p > 0$. This ordered graph has no induced monotone path of length 5, its maximum degree satisfies $\Delta(G) < (1/2 + p)n$, but the maximum size of a bi-clique in its complement is $O_p(\log n)$. Consequently, for the constant appearing in Theorem 1, we have $c_5 \leq 1/2$.

The assumption that $G_<$ contains no induced P_3 is equivalent to the property that $G_<$ is a comparability graph. In this special case (that is, for $k = 3$), Theorem 1 was established by Fox, Pach, and Tóth [**7**], and in a weaker form by Fox [**5**]. Apart from the value of the constant c, the bound is best possible for $k = 3$ and, hence, for every $k \geq 3$.

An *ordered matching* is an ordered graph on $2k$ vertices which consists of k edges, no two of which share an endpoint. Our next result is an analogue of Theorem 1 for ordered graphs that contain no induced subgraph isomorphic to a fixed ordered matching.

Theorem 2. *For every ordered matching M, there exists a constant $c = c(M) > 0$ such the following statement is true. Let $G_<$ be an ordered graph on n vertices which satisfies $\Delta(G_<) < cn$ and does not have any induced ordered subgraph isomorphic to M.*

Then the complement of $G_<$ contains a bi-clique of size at least cn.

The conclusion of Theorem 2 is stronger than that of Theorem 1: in this case we can find a linear-sized bi-clique in the complement of $G_<$.

For *unordered* graphs without (unordered) induced paths of length k, the size of the largest bi-clique that can be found in $\overline{G}$ is larger than what was shown in Theorem 1: it is linear in n. More precisely, Bousquet, Lagoutte, and Thomassé [**1**] proved that for every positive integer k, there exists $c(k) > 0$ such that, if G is an unordered graph with n vertices and at most $c(k)\binom{n}{2}$ edges, which does not have an induced path of length k, then its complement $\overline{G}$ contains a bi-clique of size at least $c(k)n$. Recently, Chudnovsky, Scott, Seymour, and Spirkl [**3**] generalized this result to any forbidden forest, instead of a path. In an upcoming work, we obtain similar extensions of Theorems 1 and 2 to other *ordered forests*.

2. Intersection graphs of curves

Given a family of sets, $\mathcal{C}$, the *intersection graph* of $\mathcal{C}$ is the graph, whose vertices correspond to the elements of $\mathcal{C}$, and two vertices are joined by an edge if and only if the corresponding sets have a nonempty intersection. A *curve* is the image of a continuous function $\phi\colon [0,1] \to \mathbb{R}^2$. A curve is said to be *x-monotone* if every vertical line intersects it in at most one point. Note that any convex set can be approximated arbitrarily closely by x-monotone curves, so the notion of x-monotone curve extends the notion of convex sets. Throughout this paper, a curve will be called a *grounded* if one of its endpoints lies on the y-axis (on the vertical line $\{x = 0\}$) and the whole curve is contained in the nonnegative half-plane $\{x \geq 0\}$. (By slight abuse of notation, we write $\{x \geq 0\}$ for the set $\{(x, y) \in \mathbb{R}^2 : x \geq 0\}$.)

It turns out that ordered graphs without certain forbidden induced ordered subgraphs capture many interesting properties of intersection graphs of curves. We will apply Theorems 1 and 2 to give a simple combinatorial proof for the following Ramsey-type result of Fox and Pach [**6**], which is related to a celebrated conjecture of Erdős and Hajnal [**4, 2**].

Theorem 3 ([**6**]). *There exists an absolute constant $c > 0$ with the following property. The intersection graph G of any collection of n x-monotone curves contains a bi-clique of size at least $cn/\log n$, or its complement $\overline{G}$ contains a bi-clique of size at least cn.*

This result is tight, up to the value of c; see [**10**]. Indeed, Fox [**5**] proved that for any $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that for every $n \in \mathbb{N}$, there exists an incomparability graph G on n vertices such that G does not contain a bi-clique of size $c(\varepsilon)n/\log n$, and the complement of G does not contain a bi-clique of size n^{ϵ}. On the other hand, every incomparability graph is isomorphic to the intersection graph of a collection of x-monotone curves [**11, 9, 10**].

It was shown in [**7**] that if the intersection graph of n x-monotone curves has at most $12^{-8}\binom{n}{2}$ edges, then the second option holds in Theorem 3: $\overline{G}$ contains a bi-clique of size at least cn. Also, the same result (with different constants) follows from a separator theorem of Lee [**8**] for string graphs. None of these arguments leave much room for replacing 12^{-8} by a decent constant. Tomon [**12**] applied some properties of partially ordered sets to establish the upper bound $\left(\frac{1}{16} - o(1)\right)\binom{n}{2}$. Somewhat surprisingly, using ordered graphs, one can precisely determine the best constant for which the statement still holds.

Theorem 4. *For any $\epsilon > 0$, there are constants $c_1 = c_1(\epsilon), c_2 = c_2(\epsilon) > 0$, and an integer $n_0 = n_0(\epsilon)$ such that the following statements are true. For every $n \geq n_0$,*

(1) *there exist n x-monotone curves such that their intersection graph G has at most $(\frac{1}{4}+\epsilon)\binom{n}{2}$ edges, but the complement of G does not contain a bi-clique of size $c_1 \log n$;*
(2) *for any n x-monotone curves such that their intersection graph G has at most $(\frac{1}{4}-\epsilon)\binom{n}{2}$ edges, the complement of G contains a bi-clique of size $c_2 n$.*

It is easy to see that every intersection graph of *convex sets* in the plane is also an intersection graph of x-monotone curves. We prove (1) by constructing n convex sets in the plane whose intersection graphs meets the requirements. Therefore, $\frac{1}{4}\binom{n}{2}$ is also a threshold for the emergence of linear sized bi-cliques in the complements of intersection graphs of convex sets.

In [**6**], Theorem 3 was established in a more general setting: without assuming that the curves are x-monotone. It is a serious challenge to extend our proof to that case. We still believe that Theorem 4 should also generalize to arbitrary curves.

Conjecture 5. For any $\epsilon > 0$, there exist $c_0 = c_0(\epsilon) > 0$ and $n_0 = n_0(\epsilon)$ with the property that for any collection of $n \geq n_0$ curves whose intersection graph has at most $(\frac{1}{4}-\epsilon)\binom{n}{2}$ edges, the complement of G contains a bi-clique of size $c_0 n$.

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