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OPERATIONAL RESULTS ON BI-ORTHOGONAL HERMITE FUNCTIONS

C. CESARANO, C. FORNARO AND L. VAZQUEZ

Abstract. By starting from the concept of the orthogonality property related to the ordinary and generalized two-variable Hermite polynomials, we present some interesting results on the class of bi-orthogonal Hermite functions.

The structure of these bi-orthogonal functions is based on the family of the two-index, two-variable Hermite polynomials of type $H_{m,n}(x,y)$ and their adjoint $G_{m,n}(x,y)$.

Many of the results presented in the first two sections are well known in literature, but the scheme used is functional to the presentation of the bi-orthogonal Hermite functions discussed in section III. The exposition of the properties satisfied by the functions $\overline{H}_{m,n}(x,y)$ and $\overline{G}_{m,n}(x,y)$ is arranged in a non-ordinary way: in fact, we deduce many relations by using the structure of the two-index, two-variable Hermite polynomials, comparing them with the known properties of the ordinary Hermite.

We also discuss a differential representation of the operators acting on the above bi-orthogonal Hermite functions and we derive some operational identities to better clarify the role of these Hermite functions.

I. Orthogonal Hermite functions of one variable

In this first section, we present some noted relations involving Hermite polynomials of one-variable and we describe their orthogonal properties to introduce the related Hermite functions. The presented identities are well known in literature and also the techniques used to prove many of the results that we remind in the following were described in the past as we cite in bibliography. So that, in this section, we remind the tools to introduce the generalizations in the following sections.

It is well known that the one-variable ordinary Hermite polynomials $\mathrm{He}_m(x)$ have the following explicit form (see [**1, 2**])

$$\mathrm{He}_m(x) = m! \sum_{r=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^r x^{n-2r}}{r!(n-2r)!2^r}. \tag{1}$$

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It is also worth to remind that the generating function of the Hermite polynomials $\mathrm{He}_m(x)$ reads (see [**3, 1, 4**])

$$\exp\left(xt - t^2\right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!}\,\mathrm{He}_n(x) \tag{2}$$

since polynomials solve the following differential difference equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} D_n(x) &= n D_{n-1}(x), \\ D_n(0) &= \frac{n!(-1)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)! 2^{\frac{n}{2}}}, \end{aligned} \tag{3}$$

where n is even.

A characteristic property satisfied by the ordinary Hermite polynomials is the orthogonality. By using this important aspect, it is possible to introduce the related Hermite functions to derive many other relations involving the Hermite polynomials of the type $\mathrm{He}_m(x)$. We start to prove an important identity for the ordinary Hermite polynomials.

Proposition I.1. *The ordinary Hermite polynomials* $\mathrm{He}_m(x)$ *satisfy the following Rodrigues formula (see* [**3**]*)*

$$\mathrm{He}_n(x) = (-1)^n \mathrm{e}^{\frac{x^2}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\mathrm{e}^{-\frac{x^2}{2}}\right).$$

Proof. By starting from the generating function relation, presented above, we can manipulate the argument of the exponential to obtain

$$\mathrm{e}^{-\frac{x^2}{2} + xt - \frac{t^2}{2}}\,\mathrm{e}^{\frac{x^2}{2}} = \sum_{n=0}^{+\infty} \frac{t^n}{n!}\,\mathrm{He}_n(x) \tag{4}$$

and then,

$$\mathrm{e}^{-\frac{1}{2}(x-t)^2} = \mathrm{e}^{-\frac{x^2}{2}} \sum_{n=0}^{+\infty} \frac{t^n}{n!}\,\mathrm{He}_n(x). \tag{5}$$

The shift operator for a function $f(x)$, which is analytic in a neighborhood of the origin, acts in the following way

$$\mathrm{e}^{\lambda \frac{\mathrm{d}}{\mathrm{d}x}} f(x) = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} f^{(n)}(x) = f(x+\lambda), \tag{6}$$

where λ is a real number and $f(x)$ is also analytic in $x + \lambda$ without any other restrictions.

After the above considerations, we can recast the l.h.s. of the relation (5) in the form

$$\mathrm{e}^{-\frac{1}{2}(x-t)^2} = \mathrm{e}^{-t\frac{\mathrm{d}}{\mathrm{d}x}} \left(\mathrm{e}^{-\frac{x^2}{2}}\right) \tag{7}$$

and then, we have

$$\mathrm{e}^{\frac{x^2}{2}}\,\mathrm{e}^{-t\frac{\mathrm{d}}{\mathrm{d}x}}\left(\mathrm{e}^{-\frac{x^2}{2}}\right)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}\,\mathrm{He}_n(x). \tag{8}$$

The exponential operator in the previous equation can be exploited to obtain

$$\mathrm{e}^{\frac{x^2}{2}}\left[\sum_{n=0}^{+\infty}\frac{(-1)^n t^n}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\right]\left(\mathrm{e}^{-\frac{x^2}{2}}\right)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}\,\mathrm{He}_n(x). \tag{9}$$

At this point, we note that the terms acting on the exponential function $\mathrm{e}^{-\frac{x^2}{2}}$ give only an operational contribute except for the term $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n$; we can rewrite the previous equation in the following more convenient form

$$\mathrm{e}^{\frac{x^2}{2}}\sum_{n=0}^{+\infty}\frac{(-1)^n t^n}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(\mathrm{e}^{-\frac{x^2}{2}}\right)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}\,\mathrm{He}_n(x) \tag{10}$$

and by equating the terms of the same power of n, we immediately obtain the thesis of the proposition, that is the Rodrigues formula. □

Since the orthogonal polynomials are defined through a weight function and determined to less than a constant, we can now investigate the properties and the related relations of the Hermite polynomials under the point of view of their orthogonality.

Proposition I.2. *The ordinary Hermite polynomials are orthogonal on the interval* $(-\infty,+\infty)$ *with respect to the weight function*

$$\mathrm{e}^{-\frac{x^2}{2}}, \tag{11}$$

that is

$$\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{He}_n(x)\,\mathrm{He}_m(x)(x)\mathrm{d}x=n!\sqrt{2\pi}\delta_{n,m}. \tag{12}$$

Proof. By using the Rodrigues Formula, we can recast the integral of the statement in the form

$$\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{He}_n(x)\,\mathrm{He}_m(x)(x)\mathrm{d}x=(-1)^n\int_{-\infty}^{+\infty}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(\mathrm{e}^{-\frac{x^2}{2}}\right)\mathrm{He}_m(x)(x)\mathrm{d}x. \tag{13}$$

By solving the integral on the r.h.s. of the above equation and using the method by parts, we get

$$\begin{aligned}&(-1)^n\int_{-\infty}^{+\infty}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\mathrm{e}^{-\frac{x^2}{2}}\right)\mathrm{He}_m(x)\mathrm{d}x\\&=(-1)^n\left\{\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\mathrm{e}^{-\frac{x^2}{2}}\right)\mathrm{He}_m(x)-\int_{-\infty}^{+\infty}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\mathrm{e}^{-\frac{x^2}{2}}\right)\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{He}_m(x)\mathrm{d}x\right]_{-\infty}^{+\infty}\right\}\end{aligned} \tag{14}$$

and then,

$$(-1)^n \int\limits_{-\infty}^{+\infty} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\mathrm{e}^{-\frac{x^2}{2}}\right) \mathrm{He}_m(x)\mathrm{d}x$$

$$= (-1)^n \Bigg\{ \lim_{\substack{a\to-\infty \\ b\to+\infty}} \left[\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1} \left(\mathrm{e}^{-\frac{x^2}{2}}\right) \mathrm{He}_m(x)\right]_a^b \tag{15}$$

$$- (-1)\Bigg[- \int\limits_{-\infty}^{+\infty} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1} \left(\mathrm{e}^{-\frac{x^2}{2}}\right) \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{He}_m(x)\mathrm{d}x\Bigg]\Bigg\}.$$

By noting that the limit in the r.h.s. of the previous relation gives zero and using the recurrence relation satisfied by the ordinary Hermite polynomials (see[**2, 5**])

$$\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{He}_n(x) = n\,\mathrm{He}_{n-1}(x) \tag{16}$$

we can obtain the expression

$$\int\limits_{-\infty}^{+\infty} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{He}_n(x)\,\mathrm{He}_m(x)(x)\mathrm{d}x = (-1)^{n+1} m! \int\limits_{-\infty}^{+\infty} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m} \left(\mathrm{e}^{-\frac{x^2}{2}}\right) \mathrm{d}x. \tag{17}$$

Regarding the integral on the r.h.s. of the above relation, we note that

$$\int\limits_{-\infty}^{+\infty} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{s} \left(\mathrm{e}^{-\frac{x^2}{2}}\right) \mathrm{d}x = 0 \tag{18}$$

after setting $n - m = s$, assuming $n \neq m$. In case $n = m$, we have

$$\int\limits_{-\infty}^{+\infty} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x = \sqrt{2\pi} \tag{19}$$

and then, the proposition is completely proved. The orthogonality property satisfied by the Hermite polynomials $\mathrm{He}_m(x)$ suggests us to introduce a family of functions, based on the Hermite polynomials themselves in such a way as similar properties are derived. □

Definition I.1. Let the ordinary Hermite polynomials be of the type $\mathrm{He}_m(x)$, we call one-variable Hermite function, the function defined by the following relation

$$\mathrm{he}_n(x) = \left(\frac{1}{\sqrt{2\pi} n!}\right)^{\frac{1}{2}} \mathrm{He}_n(x)\mathrm{e}^{-\frac{x^2}{4}}. \tag{20}$$

Proposition I.3. *The one-variable Hermite functions of the type* $\mathrm{he}_m(x)$ *are orthonormal on the interval* $(-\infty, +\infty)$*, that is*

$$\int\limits_{-\infty}^{+\infty} \mathrm{he}_n(x)\,\mathrm{he}_m(x)\mathrm{d}x = \delta_{n,m} \tag{21}$$

Proof. By substituting the explicit form of the Hermite functions $\mathrm{he}_m(x)$ in the integral, we get

$$\int_{-\infty}^{+\infty} \mathrm{he}_n(x)\,\mathrm{he}_m(x)\mathrm{d}x = \int_{-\infty}^{+\infty} \Big(\frac{1}{\sqrt{2\pi n!}}\Big)\Big(\frac{1}{\sqrt{2\pi m!}}\Big)\,\mathrm{e}^{-\frac{x^2}{4}}\,\mathrm{e}^{-\frac{x^2}{4}}\,\mathrm{He}_n(x)\,\mathrm{He}_m(x)\mathrm{d}x$$

$$= \Big(\frac{1}{2\pi n!m!}\Big)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e\,\mathrm{He}_n(x)\,\mathrm{He}_m(x)\mathrm{d}x. \tag{22}$$

Since the Hermite polynomials $\mathrm{He}_m(x)$ are orthogonal on the interval $(-\infty, +\infty)$ with the weight function $\mathrm{e}^{-\frac{x^2}{2}}$ (see Proposition I.2), we obtain

$$\int_{-\infty}^{+\infty} \mathrm{he}_n(x)\,\mathrm{he}_m(x)\mathrm{d}x = \Big(\frac{1}{n!m!}\Big)^{\frac{1}{2}} \frac{1}{\sqrt{2\pi}} n!\sqrt{2\pi}\delta_{n,m} = \sqrt{\frac{n!}{m!}}\delta_{n,m} \tag{23}$$

and then, the thesis follows immediately. □

Proposition I.4. *The one-variable orthogonal Hermite functions* $\mathrm{he}_m(x)$ *satisfy the following recurrence relations*

$$2\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{he}_n(x) = \sqrt{n}\,\mathrm{he}_{n-1}(x) - \sqrt{n+1}\,\mathrm{he}_{n+1}(x), \tag{24}$$

$$x\,\mathrm{he}_n(x) = \sqrt{n}\,\mathrm{he}_{n-1}(x) + \sqrt{n+1}\,\mathrm{he}_{n+1}(x). \tag{25}$$

Proof. By deriving both sides of definition I.1 with respect to x, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{he}_n(x) = \Big(\frac{1}{\sqrt{2\pi n!}}\Big)^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}x}\Big(\mathrm{He}_n(x)\mathrm{e}^{-\frac{x^2}{4}}\Big) \tag{26}$$

and using the recurrence relation (16) showed in Proposition I.2, we can write the above equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{he}_n(x) = \Big(\frac{1}{\sqrt{2\pi n!}}\Big)^{\frac{1}{2}} \Big[n\,\mathrm{He}_{n-1}(x)\mathrm{e}^{-\frac{x^2}{4}} - \frac{x}{2}\,\mathrm{He}_n(x)\mathrm{e}^{-\frac{x^2}{4}}\Big]. \tag{27}$$

It is easy to prove that the ordinary Hermite polynomial $\mathrm{He}_m(x)$ satisfies the further relation

$$\mathrm{He}_n(x) = \frac{1}{x}\,[n\,\mathrm{He}_{n-1}(x) + \mathrm{He}_{n+1}(x)] \tag{28}$$

and then, we can obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{he}_n(x)\\ &= \Big(\frac{1}{\sqrt{2\pi n!}}\Big)^{\frac{1}{2}} \Big[n\,\mathrm{He}_{n-1}(x)\mathrm{e}^{-\frac{x^2}{4}} - \Big(\frac{1}{2}\mathrm{e}^{-\frac{x^2}{4}}\Big(n\,\mathrm{He}_{n-1}(x) + \mathrm{He}_{n+1}(x)\Big)\Big)\Big].\end{aligned} \tag{29}$$

By substituting the expression of Hermite polynomials $\mathrm{He}_{n-1}(x)$ and $\mathrm{He}_{n+1}(x)$ in terms of the orthogonal Hermite functions $\mathrm{he}_m(x)$ (see Definition I.1), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{he}_n(x) = \Big(\frac{1}{\sqrt{2\pi}n!}\Big)^{\frac{1}{2}}\Big[& n\mathrm{e}^{-\frac{x^2}{4}}\Big(\sqrt{2\pi}(n-1)!\Big)^{\frac{1}{2}}\mathrm{e}^{\frac{x^2}{4}}\,\mathrm{he}_{n-1}(x)\\ &-\frac{1}{2}\mathrm{e}^{-\frac{x^2}{4}}n\Big(\sqrt{2\pi}(n-1)!\Big)^{\frac{1}{2}}\mathrm{e}^{\frac{x^2}{4}}\,\mathrm{he}_{n-1}(x)\\ &-\frac{1}{2}\mathrm{e}^{-\frac{x^2}{4}}\Big(\sqrt{2\pi}(n+1)!\Big)^{\frac{1}{2}}\mathrm{e}^{\frac{x^2}{4}}\,\mathrm{he}_{n+1}(x)\Big]\end{aligned}\tag{30}$$

and then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{he}_n(x) = &\frac{n[(n-1)!]^{\frac{1}{2}}}{(n!)^{\frac{1}{2}}}\,\mathrm{he}_{n-1}(x)\\ &-\frac{1}{2}\frac{n[(n-1)!]^{\frac{1}{2}}}{(n!)^{\frac{1}{2}}}\,\mathrm{he}_{n-1}(x)-\frac{1}{2}\frac{n[(n+1)!]^{\frac{1}{2}}}{(n!)^{\frac{1}{2}}}\,\mathrm{he}_{n+1}(x),\end{aligned}\tag{31}$$

which proves the first relation of the statement. To prove the proposition completely, we start with a note that the recurrence relation (28) verified from the Hermite polynomials $\mathrm{He}_m(x)$ can be recast in the form

$$x\,\mathrm{He}_n(x) = \mathrm{He}_{n+1}(x) + n\,\mathrm{He}_{n-1}(x).\tag{32}$$

By substituting the expressions of the ordinary Hermite polynomials in terms of the related Hermite functions, we immediately obtain

$$\begin{aligned}&x\Big[\Big(\sqrt{2\pi}n!\Big)^{\frac{1}{2}}\mathrm{e}^{\frac{x^2}{4}}\,\mathrm{he}_n(x)\Big],\\ &=\Big(\sqrt{2\pi}(n+1)!\Big)^{\frac{1}{2}}\mathrm{e}^{\frac{x^2}{4}}\,\mathrm{he}_{n+1}(x)+n\Big(\sqrt{2\pi}(n-1)!\Big)^{\frac{1}{2}}\mathrm{e}^{\frac{x^2}{4}}\,\mathrm{he}_{n-1}(x),\end{aligned}\tag{33}$$

which finally proves the second recurrence relations. □

II. Differential relations involving orthogonal Hermite functions

In this section, we follow the same approach of the previous section to present some interesting relations involved the generalized two-variable Hermite polynomials and we also present the related Hermite functions.

By using the concept and related formalism of the shift operators regarding the ordinary Hermite polynomials (see [**6, 7**]), we can explore the differential characteristics involving the orthogonal Hermite functions of type $\mathrm{he}_m(x)$.

It is easy to note that

$$\Big(\frac{\mathrm{d}}{\mathrm{d}x}+\frac{x}{2}\Big)\mathrm{he}_n(x) = \sqrt{n}\,\mathrm{he}_{n-1}(x)\tag{34}$$

$$\Big(-\frac{\mathrm{d}}{\mathrm{d}x}+\frac{x}{2}\Big)\mathrm{he}_n(x) = \sqrt{n+1}\,\mathrm{he}_{n+1}(x)\tag{35}$$

which show the action on the Hermite function. Then, by setting

$$\hat{a}_- = \left(\tfrac{\mathrm{d}}{\mathrm{d}x} + \tfrac{x}{2}\right), \qquad \hat{a}_+ = \left(-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x}{2}\right), \tag{36}$$

we can express and rewritethe previous relations in the formal way

$$\begin{aligned} \hat{a}_- \,\mathrm{he}_n(x) &= \sqrt{n}\,\mathrm{he}_{n-1}(x), \\ \hat{a}_+ \,\mathrm{he}_n(x) &= \sqrt{n+1}\,\mathrm{he}_{n+1}(x). \end{aligned} \tag{37}$$

The shift operators related to Hermite polynomials are dependent on discrete parameters while the above operators keep the same expression, that is they do not change with the index function. It is worth to note that the following relation holds

$$\hat{a}_+\hat{a}_- \,\mathrm{he}_n(x) = n\,\mathrm{he}_n(x), \tag{38}$$

which can be used to state the result.

Theorem II.1. *The one-variable orthogonal Hermite functions* $\mathrm{he}_m(x)$ *solve the following ordinary differential equations*

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{x^2}{4} + \left(n + \frac{1}{2}\right)\right] \mathrm{he}_n(x) = 0. \tag{39}$$

Proof. By using the operatorial relation (38), we have

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x}{2}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x}{2}\right) \mathrm{he}_n(x) = n\,\mathrm{he}_n(x) \tag{40}$$

and then,

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{x}{2}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2} + \frac{x}{2}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x^2}{4}\right) \mathrm{he}_n(x) = n\,\mathrm{he}_n(x), \tag{41}$$

and finally,

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{4} - \frac{1}{2} - n\right) \mathrm{he}_n(x) = 0, \tag{42}$$

which completely proves the statement of theorem. □

At the beginning of this paper, we have presented the generating function of the Hermite polynomial $\mathrm{He}_m(x)$ and in Definition I.1, we have introduced the orthogonal Hermite function $\mathrm{he}_m(x)$ based on the ordinary Hermite polynomials. It is now possible to derive the generating function for this type of Hermite functions by manipulating relations (2) and (20). In fact, we have

$$\exp\left(xt - \frac{t^2}{2}\right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!}\left(\sqrt{2\pi} n!\right)^{\frac{1}{2}} \mathrm{e}^{\frac{x^2}{4}}\, \mathrm{he}_n(x) \tag{43}$$

which immediately gives the link between the Hermite function $\mathrm{he}_m(x)$ and its generating function, that is

$$\frac{1}{\left(\sqrt{2\pi}\right)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-t)^2 - \frac{x^2}{4}\right) = \sum_{n=0}^{+\infty} \frac{t^n}{(n!)^{\frac{1}{2}}}\, \mathrm{he}_n(x)\mathrm{e}^{-\frac{x^2}{2}}. \tag{44}$$

In the same way we can derive the analogous Rodrigues formula for the orthogonal Hermite functions. In fact, by substituting in the Rodrigues formula, related to the ordinary Hermite polynomials $\mathrm{He}_m(x)$ stated in Proposition I.1, we get the expression of the polynomials $\mathrm{He}_m(x)$ in terms of the functions $\mathrm{he}_m(x)$

$$\left(\sqrt{2\pi} n!\right)^{\frac{1}{2}} \mathrm{e}^{\frac{x^2}{4}} \,\mathrm{he}_n(x) = (-1)^n \mathrm{e}^{\frac{x^2}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\mathrm{e}^{-\frac{x^2}{2}}\right) \tag{45}$$

and by rearranging the terms, we end up with the expression

$$\mathrm{he}_n(x) = \frac{1}{\left(\sqrt{2\pi}\right)^{\frac{1}{2}}} (-1)^n \frac{1}{(n!)^{\frac{1}{2}}} \mathrm{e}^{\frac{x^2}{4}} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\mathrm{e}^{-\frac{x^2}{2}}\right) \tag{46}$$

that represents the Rodrigues formula for the orthogonal Hermite functions $\mathrm{he}_m(x)$. $\blacksquare$

We can now generalize the above results obtained by using the ordinary one-variable Hermite polynomials and applying them to the two-variable Hermite polynomials family. We remind that the explicit forms of the generalized Hermite polynomials of type $H_m(x, y)$ and $\mathrm{He}_m(x)$ read (see [**1**])

$$H_m(x, y) = \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{m!}{n!(m-2n)!} y^n x^{m-2n} \tag{47}$$

and

$$\mathrm{He}_m(x, y) = \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{m!}{n!(m-2n)!} (-y)^n (2x)^{m-2n}. \tag{48}$$

Since we have introduced the orthogonal Hermite functions of one-variable by using the structure and the properties of the ordinary Hermite polynomials $\mathrm{He}_m(x)$, we expect that it is also possible to define analogous Hermite functions of two variables that are orthogonal by using the expression of the generalized two-variable Hermite polynomials. This is obviously possible, but we face the question starting directly by the definition of the one-variable Hermite functions $\mathrm{he}_m(x)$.

Definition II.1. Let x and y be two real variables and $\mathrm{he}_m(x)$ be the one-variable Hermite function. We define the two-variable Hermite function $\mathrm{he}_m(x, y)$ given by the expression

$$\mathrm{he}_n(x, y) = \sum_{r=0}^{[n/2]} \sqrt{\frac{n!}{(n-2r)!r!}} \,\mathrm{he}_{n-2r}(x) \,\mathrm{he}_r(y). \tag{49}$$

Theorem II.2. *The two-variable Hermite functions* $\mathrm{he}_m(x, y)$ *are orthogonal functions on the interval* $(-\infty, +\infty) \times (-\infty, +\infty)$.

Proof. We have to prove that the following integral

$$\int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{he}_n(x, y) \,\mathrm{he}_m(x, y) \mathrm{d}x \tag{50}$$

is a finite number. By substituting the explicit expression of the two-variable Hermite functions $\mathrm{he}_m(x,y)$ given in Definition II.1, we get

$$\int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{he}_n(x,y)\,\mathrm{he}_m(x,y)\mathrm{d}x = \sum_{r=0}^{[n/2]}\sum_{s=0}^{[m/2]} \sqrt{\frac{n!m!}{(n-2r)!(m-2s)!r!s!}}$$
$$\times \int_{-\infty}^{+\infty} \mathrm{he}_{n-2r}(x)\,\mathrm{he}_{m-2s}(x)\mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{he}_r(y)\,\mathrm{he}_{m-2s}(y)\mathrm{d}y. \tag{51}$$

Since the one-variable Hermite functions are orthonormal on the interval $(-\infty,+\infty)$ (see eq. (21)), we have

$$\int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{he}_n(x,y)\,\mathrm{he}_m(x,y)\mathrm{d}x$$
$$= \sum_{r=0}^{[n/2]}\sum_{s=0}^{[m/2]} \sqrt{\frac{n!m!}{(n-2r)!(m-2s)!r!s!}} \int_{-\infty}^{+\infty} \mathrm{he}_{n-2r}(x)\,\mathrm{he}_{m-2s}(x)\mathrm{d}x\delta_{r,s}. \tag{52}$$

We note that in the above summations, all the addends are zero except $r = s$. Then we can rewrite the previous relation in the form

$$\int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{he}_n(x,y)\,\mathrm{he}_m(x,y)\mathrm{d}x$$
$$= \sum_{r=0}^{[n/2]}\sum_{r=0}^{[m/2]} \sqrt{\frac{n!m!}{((n-2r)!)^2(r!)^2}} \int_{-\infty}^{+\infty} \mathrm{he}_{n-2r}(x)\,\mathrm{he}_{m-2r}(x)\mathrm{d}x \tag{53}$$

and by applying again the orthonormal property of the one-variable Hermite functions $\mathrm{he}_m(x)$, we similarly obtain

$$\int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{he}_n(x,y)\,\mathrm{he}_m(x,y)\mathrm{d}x = \sum_{r=0}^{[n/2]}\sum_{r=0}^{[m/2]} \sqrt{\frac{n!m!}{((n-2r)!)^2(r!)^2}}\delta_{n,m}. \tag{54}$$

Also in this case, the only not zero value is obtained for $n = m$, so we can conclude

$$\int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{he}_n(x,y)\,\mathrm{he}_m(x,y)\mathrm{d}x = \sum_{r=0}^{[n/2]} \frac{n!}{(n-2r)!r!}, \tag{55}$$

which proves the orthogonality of the two-variable Hermite functions $\mathrm{he}_m(x,y)$. It could be useful to observe that the term obtained in the proof of Theorem II.2 can be read as a special case of the two-variable Hermite polynomials of the type $H_m(x,y)$

$$\mathrm{H}_n\left(\frac{1}{2},-1\right) = \sum_{r=0}^{[n/2]} \frac{n!}{(n-2r)!r!}. \tag{56}$$

We can derive the generating function for the two-variable orthogonal Hermite functions $\mathrm{he}_m(x,y)$ by using the structure and the identities of the Hermite polynomials. It is known that the generating function of the two-variable Hermite polynomials of type $\mathrm{He}_m(x,y)$ can be written in the form (see [**6, 2**])

$$\mathrm{H}_n^{(m)}(x,y) = \mathrm{e}^{y \frac{\partial^m}{\partial x^m}}\, x^n. \tag{57}$$

By manipulating the argument of the exponential, we obtain

$$\begin{aligned}\exp\left(xt - \frac{t^2}{2} + yt^2 - \frac{t^4}{2}\right) &= \exp\left(xt - \frac{t^2}{2}\right)\exp\left(yt^2 - \frac{t^4}{2}\right)\\ &= \sum_{m=0}^{+\infty} \frac{t^m}{m!}\,\mathrm{He}_m(x)(x) \sum_{r=0}^{+\infty} \frac{t^{2r}}{r!}\,\mathrm{He}_r(y).\end{aligned} \tag{58}$$

By setting $m + 2r = n$ after rearranging the indices in the above summations, we end up with

$$\mathrm{He}_n(x,y) = n! \sum_{r=0}^{[n/2]} \frac{1}{(n-2r)!r!}\,\mathrm{He}_{n-2r}(x)\,\mathrm{He}_r(y) \tag{59}$$

which gives an expression of the two-variable Hermite polynomials $\mathrm{He}_m(x,y)$ in terms of the ordinary one-variable Hermite polynomials. Further we use the relation showed above to state the link between the two-variable orthogonal Hermite functions $\mathrm{he}_m(x,y)$ and their generating function.

In the definition of the functions $\mathrm{he}_m(x,y)$ (see equation (49)) we start to substitute the expression of the one-variable orthogonal Hermite functions $\mathrm{he}_m(x)$ given in Definition I.1

$$\begin{aligned}\mathrm{he}_n(x,y) = &\sum_{r=0}^{[n/2]} \sqrt{\frac{n!}{(n-2r)!r!}} \left(\frac{1}{\sqrt{2\pi}(n-2r)!}\right)^{\frac{1}{2}} \left(\frac{1}{\sqrt{2\pi}r!}\right)^{\frac{1}{2}}\\ &\times \mathrm{e}^{-\frac{x^2}{4}}\,\mathrm{e}^{-\frac{y^2}{4}}\,\mathrm{He}_{n-2r}(x)\,\mathrm{He}_r(y)\end{aligned} \tag{60}$$

which gives

$$\mathrm{he}_n(x,y) = \mathrm{e}^{-\frac{x^2}{4}}\,\mathrm{e}^{-\frac{y^2}{4}}\,\frac{1}{\sqrt{2\pi}}\sqrt{n!} \sum_{r=0}^{[n/2]} \frac{1}{(n-2r)!r!}\,\mathrm{He}_{n-2r}(x)\,\mathrm{He}_r(y). \tag{61}$$

By substituting the expression stated above, we have

$$\mathrm{he}_n(x,y) = \frac{\sqrt{n!}}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{4}}\,\mathrm{e}^{-\frac{y^2}{4}}\,\frac{\mathrm{He}_n(x,y)}{n!}. \tag{62}$$

By expressing the two-variable Hermite polynomials $\mathrm{He}_m(x,y)$ in terms of the Hermite functions $\mathrm{he}_m(x,y)$, the previous equation reads

$$\mathrm{He}_n(x,y) = \frac{n!}{\sqrt{n!}}\sqrt{2\pi}\mathrm{e}^{\frac{x^2+y^2}{4}}\,\mathrm{he}_n(x,y). \tag{63}$$

By inserting in expression (58), we get

$$\exp\Big(xt-\frac{t^2}{2}+yt^2-\frac{t^4}{2}\Big)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}\Big[\frac{n!}{\sqrt{n!}}\sqrt{2\pi}\mathrm{e}^{\frac{x^2+y^2}{4}}\,\mathrm{he}_m(x,y)\Big] \tag{64}$$

and then, we can finally state the expression of the generating function of the two-variable Hermite functions $\mathrm{he}_m(x,y)$

$$\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}(x-t)-\frac{1}{2}(y-t^2)}\,\mathrm{e}^{\frac{x^2+y}{4}}=\sum_{n=0}^{+\infty}\frac{t^n}{\sqrt{n!}}\,\mathrm{he}_n(x,y). \tag{65}$$

The two-variable Hermite polynomials satisfy many interesting identities (see [**5, 6**]) that allow to derive similar relations for the two-variable Hermite functions $\mathrm{he}_m(x,y)$. The starting point is the link between the polynomials $\mathrm{He}_m(x,y)$ and the functions $\mathrm{he}_m(x,y)$ showed above.

By deriving with respect to x in relation (62), we have

$$\frac{\partial}{\partial x}\,\mathrm{he}_n(x,y)=\frac{1}{\sqrt{2\pi n!}}\,\mathrm{e}^{-\frac{y^2}{4}}\Big[-\frac{x}{2}\,\mathrm{e}^{-\frac{x^2}{4}}\,\mathrm{He}_n(x,y)+\mathrm{e}^{-\frac{x^2}{4}}\,\frac{\partial}{\partial x}\,\mathrm{He}_n(x,y)\Big] \tag{66}$$

and then,

$$\frac{\partial}{\partial x}\mathrm{he}_n(x,y)=-\frac{1}{\sqrt{2\pi n!}}\frac{x}{2}\mathrm{e}^{-\frac{x^2+y^2}{4}}\mathrm{He}_n(x,y)+\frac{1}{\sqrt{2\pi n!}}\mathrm{e}^{-\frac{x^2+y^2}{4}}n\mathrm{He}_{n-1}(x,y). \tag{67}$$

By applying the identities in equation (63), we can easily conclude with the following generalization

$$\frac{\partial}{\partial x}\,\mathrm{he}_n(x,y)=-\frac{x}{2}\,\mathrm{he}_n(x,y)+\sqrt{n}\,\mathrm{he}_{n-1}(x,y). \tag{68}$$

In the same way it is possible to state an analogous recurrence relation satisfied by the Hermite functions $\mathrm{he}_m(x,y)$. In fact, by deriving with respect to y in equation (62), we obtain

$$\frac{\partial}{\partial y}\,\mathrm{he}_n(x,y)=\frac{1}{\sqrt{2\pi n!}}\mathrm{e}^{-\frac{x^2}{4}}\Big[-\frac{y}{2}\mathrm{e}^{-\frac{x^2}{4}}\,\mathrm{He}_n(x,y)+\mathrm{e}^{-\frac{y^2}{4}}\,\frac{\partial}{\partial y}\,\mathrm{He}_n(x,y)\Big] \tag{69}$$

and by using the properties of two-variable Hermite polynomials, we can rewrite the above expression in the form

$$\begin{aligned}\frac{\partial}{\partial y}\,\mathrm{he}_n(x,y)=&-\frac{1}{\sqrt{2\pi n!}}\frac{y}{2}\,\mathrm{e}^{-\frac{x^2+y^2}{4}}\,\mathrm{He}_n(x,y)\\&+\frac{1}{\sqrt{2\pi n!}}\,\mathrm{e}^{-\frac{x^2+y^2}{4}}\,n(n-1)\,\mathrm{He}_{n-2}(x,y).\end{aligned} \tag{70}$$

By using again the identity in equation (63), we can finally state the second generalized recurrence relation for the two-variable Hermite function $\mathrm{he}_m(x,y)$

$$\frac{\partial}{\partial y}\,\mathrm{he}_n(x,y)=-\frac{y}{2}\,\mathrm{he}_n(x,y)+\sqrt{n(n-1)}\,\mathrm{he}_{n-2}(x,y). \tag{71}$$

A further recurrence relation involving the Hermite functions $\mathrm{he}_m(x,y)$ can be deduced by operating directly in the equation linking the generalized Hermite polynomials of the type $\mathrm{He}_m(x,y)$ and its generating function. We remind the reader that the generating function of the polynomials $\mathrm{He}_m(x,y)$ has the expression

$$\exp\left(xt - \frac{t^2}{2} + yt^2 - \frac{t^4}{2}\right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!}\,\mathrm{He}_n(x,y) \tag{72}$$

and by deriving both sides with respect to t, we obtain

$$(x - t + 2yt - 2t^3)\sum_{n=0}^{+\infty} \frac{t^n}{n!}\,\mathrm{He}_n(x,y) = \sum_{n=0}^{+\infty} n\frac{t^{n-1}}{n!}\,\mathrm{He}_n(x,y). \tag{73}$$

By exploiting the terms in the above relation, we can write

$$\begin{aligned} &x\sum_{n=0}^{+\infty} \frac{t^n}{n!}\,\mathrm{He}_n(x,y) - \sum_{n=0}^{+\infty} \frac{t^{n+1}}{n!}\,\mathrm{He}_n(x,y) \\ &+ 2y\sum_{n=0}^{+\infty} \frac{t^{n+1}}{n!}\,\mathrm{He}_n(x,y) - 2\sum_{n=0}^{+\infty} \frac{t^{n+3}}{n!}\,\mathrm{He}_n(x,y) = \sum_{n=0}^{+\infty} n\frac{t^{n-1}}{n!}\,\mathrm{He}_n(x,y). \end{aligned} \tag{74}$$

By equating the terms of the same power of n, we have

$$\begin{aligned} &x\frac{\mathrm{He}_n(x,y)}{n!} + (2y-1)\frac{\mathrm{He}_{n-1}(x,y)}{(n-1)!} \\ &\qquad - 2\frac{\mathrm{He}_{n-3}(x,y)}{(n-3)!} = \frac{n+1}{(n+1)!}\,\mathrm{He}_{n+1}(x,y), \end{aligned} \tag{75}$$

which gives the important recurrence relation for the generalized Hermite polynomials $\mathrm{He}_m(x,y)$

$$\begin{aligned} &x\,\mathrm{He}_n(x,y) + (2y-1)n\,\mathrm{He}_{n-1}(x,y) \\ &\qquad - 2\left[n(n-1)(n-2)\right]\mathrm{He}_{n-3}(x,y) = \mathrm{He}_{n+1}(x,y). \end{aligned} \tag{76}$$

We can use the relation stated above to derive the analogous identity for the two-variable Hermite functions $\mathrm{he}_m(x,y)$. In fact, by substituting the expression of the Hermite polynomials $\mathrm{He}_m(x,y)$ in terms of the Hermite functions $\mathrm{he}_m(x,y)$ given by equation (63), we have

$$\begin{aligned} &x\sqrt{n!}\sqrt{2\pi}\mathrm{e}^{\frac{x^2+y^2}{4}}\,\mathrm{he}_n(x,y) + (2y-1)n\sqrt{(n-1)!}\sqrt{2\pi}\mathrm{e}^{\frac{x^2+y^2}{4}}\,\mathrm{he}_{n-1}(x,y) \\ &\qquad - 2\left[n(n-1)(n-2)\right]\sqrt{(n-3)!}\sqrt{2\pi}\mathrm{e}^{\frac{x^2+y^2}{4}}\,\mathrm{he}_{n-3}(x,y) \\ &\qquad = \sqrt{(n+1)!}\sqrt{2\pi}\mathrm{e}^{\frac{x^2+y^2}{4}}\,\mathrm{he}_{n+1}(x,y). \end{aligned} \tag{77}$$

We can finally conclude

$$\begin{aligned} &x\,\mathrm{he}_n(x,y) + (2y-1)\sqrt{n}\,\mathrm{he}_{n-1}(x,y) \\ &\qquad - 2\sqrt{n(n-1)(n-2)}\,\mathrm{he}_{n-3}(x,y) = \sqrt{n+1}\,\mathrm{he}_{n+1}(x,y). \end{aligned} \tag{78}$$

□

III. Bi-Orthogonal Hermite functions

In a previous paper (see [**8**]), we presented the two-index, two-variable Hermite polynomials of the type $\mathrm{H}_{m,n}(x,y)$ and defined their associate $\mathrm{G}_{m,n}(x,y)$, by deriving many properties and interesting identities for both type of generalized vectorial polynomials. It is now interesting to explore the possibility to find similar Hermite functions as those defined in the previous sections of the present article, in order to obtain an extension of the concepts and the related identities satisfied from the Hermite polynomials $\mathrm{H}_{m,n}(x,y)$ and their associate $\mathrm{G}_{m,n}(x,y)$. The structure of the vectorial extension Hermite polynomials is based on the fact that a vector index acts on a vector variable or, that is the same, a couple of indexes act on a couple of variables. We see that many of the properties satisfied by this family of Hermite polynomials could be referred to the analogous ones satisfied by the ordinary Hermite polynomials of type $\mathrm{He}_m(x)$ and their generalizations, but the cited properties, relevant to the polynomials $\mathrm{H}_{m,n}(x,y)$ and $\mathrm{G}_{m,n}(x,y)$, are deduced without making use of $\mathrm{He}_m(x)$ properties, this means that they could not be obtained as natural extensions of those relevant to one-index Hermite polynomials.

This suggests that we can not expect the same relation linking the two-index, two-variable Hermite polynomials $\mathrm{H}_{m,n}(x,y)$ and $\mathrm{G}_{m,n}(x,y)$, and the related Hermite functions we are going to define (see [**9, 10**]). We also see that the concept of orthogonality is not the same as the existing one for the one-index Hermite polynomials of type $\mathrm{He}_m(x)$ and $\mathrm{He}_m(x,y)$. In particular, we will prove that the bi-orthogonal Hermite functions solve a partial differential equations by using the formalism and the related concepts of the shift operators which will be introduced in the description of the structure of the discussed bi-orthogonal functions.

We start indeed from this last point. We will prove that the vectorial Hermite polynomials of the type $\mathrm{H}_{m,n}(x,y)$ and their associate $\mathrm{G}_{m,n}(x,y)$ satisfy a bi-orthogonality condition instead the orthogonality condition, in the sense that the polynomials $\mathrm{H}_{m,n}(x,y)$ are orthogonal with respect to the associate polynomials $\mathrm{G}_{m,n}(x,y)$.

Theorem III.1. *The two-index, two-variable Hermite polynomials* $\mathrm{H}_{m,n}(x,y)$ *and their related associate* $\mathrm{G}_{m,n}(x,y)$ *satisfy the following bi-orthogonality condition*

$$\int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{d}x \; \mathrm{H}_{m,n}(x,y) G_{r,s}(x,y) \mathrm{e}^{-\frac{1}{2}\underline{z}^t \hat{M} \underline{z}} = \frac{2\pi}{\sqrt{\Delta}} m!n!\delta_{m,r}\delta_{n,s}, \tag{79}$$

where $\underline{z} = \binom{x}{y}$ *is a vector of space* $\mathbb{R}^2$*, and* $\hat{M} = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ *is the matrix associated to the positive definite quadratic form*

$$q(x,y) = ax^2 + 2bxy + cy^2, \quad a, c > 0, \quad \Delta = ac - b^2 > 0 \tag{80}$$

with a, b, c real numbers.

Proof. We know that the generating function of Hermite polynomials $\mathrm{H}_{m,n}(x,y)$ reads (see [**8**])

$$\mathrm{e}^{\underline{z}^t \hat{M}\underline{w} - \frac{1}{2}\underline{w}^t \hat{M}\underline{w}} = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{t^m}{m!}\frac{u^n}{n!}\,\mathrm{H}_{m,n}(x,y), \tag{81}$$

where $\underline{z} = \binom{x}{y}$ and $\underline{w} = \binom{t}{u}$ are two vectors of space $\mathbb{R}^2$. It is possible to recast the above equation in a more convenient form. By acting on the argument of the exponential, we have indeed

$$\mathrm{e}^{-\frac{1}{2}\left[(\underline{z}-\underline{w})^t \hat{M}(\underline{z}-\underline{w})\right]} = \mathrm{e}^{-\frac{1}{2}\left[\underline{z}^t \hat{M}\underline{z}\right]} \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{t^m}{m!}\frac{u^n}{n!}\mathrm{H}_{m,n}(x,y) \tag{82}$$

which better outlines the analogy between the structure of the generating functions related to the ordinary Hermite polynomials $\mathrm{He}_m(x)$ and the two-index, two-variable Hermite polynomials $\mathrm{H}_{m,n}(x,y)$. This aspect allows us to obtain a generalization of Rodrigues formula showed for the ordinary Hermite polynomials. In fact, by acting directly on the statement contained in Proposition I.2, we immediately have

$$\begin{aligned}\mathrm{H}_{m,n}(x,y) &= (-1)^{m+n}\,\mathrm{e}^{\frac{1}{2}\left(\underline{z}^t \hat{M}\underline{z}\right)}\,\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[\mathrm{e}^{-\frac{1}{2}\left(\underline{z}^t \hat{M}\underline{z}\right)}\right]\mathrm{H}_{m,n}(x,y)\\ &= (-1)^{m+n}\,\mathrm{e}^{\frac{1}{2}\left(\underline{z}^t \hat{M}\underline{z}\right)}\,\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[\mathrm{e}^{-\frac{1}{2}\left(\underline{z}^t \hat{M}\underline{z}\right)}\right]\end{aligned} \tag{83}$$

representing the Rodrigues formula related to the Hermite polynomials $\mathrm{H}_{m,n}(x,y)$.

The above identity could be recast in the form

$$\mathrm{e}^{-\frac{1}{2}\left(\underline{z}^t \hat{M}\underline{z}\right)}\mathrm{H}_{m,n}(x,y) = (-1)^{m+n}\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[\mathrm{e}^{-\frac{1}{2}\left(\underline{z}^t \hat{M}\underline{z}\right)}\right], \tag{84}$$

which allows us to rewrite the integral in the statement in a operational form

$$\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}y\Big[(-1)^{m+n}\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left(\mathrm{e}^{-\frac{1}{2}\underline{z}^t \hat{M}\underline{z}}\right)\mathrm{G}_{r,s}(x,y)\Big]. \tag{85}$$

We first start to evaluate the integral with respect to variable y

$$(-1)^{m+n}\int_{-\infty}^{+\infty}\Big[\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left(\mathrm{e}^{-\frac{1}{2}\underline{z}^t \hat{M}\underline{z}}\right)\mathrm{G}_{r,s}(x,y)\Big]\mathrm{d}y, \tag{86}$$

which integrating by parties, gives

$$\begin{aligned}&(-1)^{m+n}\int_{-\infty}^{+\infty}\Big[\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left(\mathrm{e}^{-\frac{1}{2}\underline{z}^t \hat{M}\underline{z}}\right)\mathrm{G}_{r,s}(x,y)\Big]\mathrm{d}y\\ &= (-1)^{m+(n+1)}\int_{-\infty}^{+\infty}\Big[\frac{\partial^{m+(n-1)}}{\partial x^m \partial y^{n-1}}\left(\mathrm{e}^{-\frac{1}{2}\underline{z}^t \hat{M}\underline{z}}\right)\frac{\partial}{\partial y}\mathrm{G}_{r,s}(x,y)\Big]\mathrm{d}y.\end{aligned} \tag{87}$$

By using the techniques of the vectorial derivation (see [**8**]), we note that

$$\frac{\partial}{\partial y}\sum_{r=0}^{+\infty}\sum_{s=0}^{+\infty}\frac{k^m}{r!}\frac{h^n}{s!}\mathrm{G}_{r,s}(x,y)=\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{k^m}{r!}\frac{h^n}{s!}\frac{\partial}{\partial y}\mathrm{G}_{r,s}(x,y), \tag{88}$$

that is

$$sG_{r,s-1}(x,y)=\frac{\partial}{\partial y}\mathrm{G}_{r,s}(x,y). \tag{89}$$

By substituting this last expression into the integral, we have the relation

$$\begin{aligned}&(-1)^{m+n}\int\limits_{-\infty}^{+\infty}\Big[\frac{\partial^{m+n}}{\partial x^m\partial y^n}\left(\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\right)\mathrm{G}_{r,s}(x,y)\Big]\mathrm{d}y\\&=(-1)^{m+(n+1)}s\int\limits_{-\infty}^{+\infty}\Big[\frac{\partial^{m+(n-1)}}{\partial x^m\partial y^{n-1}}\left(\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\right)\mathrm{G}_{r,s-1}(x,y)\Big]\mathrm{d}y.\end{aligned} \tag{90}$$

Without prejudicing the generality, we can suppose that $n\geq s$, and then, iterating the process on the index s, we finally obtain

$$\begin{aligned}&(-1)^{m+n}\int\limits_{-\infty}^{+\infty}\Big[\frac{\partial^{m+n}}{\partial x^m\partial y^n}\left(\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\right)\mathrm{G}_{r,s}(x,y)\Big]\mathrm{d}y\\&=(-1)^{m+(n+s)}s!\int\limits_{-\infty}^{+\infty}\Big[\frac{\partial^{m+(n-s)}}{\partial x^m\partial y^{n-s}}\left(\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\right)\mathrm{G}_{r,0}(x,y)\Big]\mathrm{d}y\end{aligned} \tag{91}$$

which is not zero if and only if $n=s$.

Let $n=s$, the double integral in the statement becomes

$$\begin{aligned}&\int\limits_{-\infty}^{+\infty}\mathrm{d}y\int\limits_{-\infty}^{+\infty}\mathrm{d}x\mathrm{H}_{m,n}(x,y)\mathrm{G}_{r,s}(x,y)\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\\&=(-1)^{m+(n-s)}n!\int\limits_{-\infty}^{+\infty}\mathrm{d}x\int\limits_{-\infty}^{+\infty}\mathrm{d}y\Big[\frac{\partial^m}{\partial x^m}\left(\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\right)\mathrm{G}_{r,0}(x,y)\Big].\end{aligned} \tag{92}$$

which once integrated by parties with respect to the variable x, gives

$$\begin{aligned}&(-1)^{m+(n+s)}n!\Big\{\int\limits_{-\infty}^{+\infty}\Big[\lim_{\substack{a\to-\infty\\b\to+\infty}}\left(\frac{\partial^{m-1}}{\partial x^{m-1}}\left(\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\right)\mathrm{G}_{r,0}(x,y)\right)\Big]_a^b\\&-(-1)\Big[-\int\limits_{-\infty}^{+\infty}\frac{\partial^{m-1}}{\partial x^{m-1}}\left(\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\right)\frac{\partial}{\partial x}\mathrm{G}_{r,0}(x,y)\Big]\mathrm{d}y\Big\}.\end{aligned} \tag{93}$$

By operating in the same way as above, regarding the partial derivative acts on the polynomial $\mathrm{G}_{r,0}(x,y)$, we have

$$\frac{\partial}{\partial x}\mathrm{G}_{r,0}(x,y) = r!\mathrm{G}_{r-1,0}(x,y), \tag{94}$$

which after substitued in the integral, gives

$$(-1)^{(m+1)+(n+s)} n!r!\Big[\int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty}\frac{\partial^{m-1}}{\partial x^{m-1}}\Big(\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\Big)\,\mathrm{G}_{r-1,0}(x,y)\mathrm{d}x\Big)\mathrm{d}y\Big]. \tag{95}$$

We can suppose $m \geq r$ and by iterating the process, it leads to the expression

$$(-1)^{(m+r)+(n+s)} n!r!\Big[\int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty}\frac{\partial^{m-r}}{\partial x^{m-r}}\Big(\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\Big)\,\mathrm{G}_{0,0}(x,y)\mathrm{d}x\Big)\mathrm{d}y\Big], \tag{96}$$

where it is easy to observe that the integral provides a zero result when m is not equal to r. By assuming $m = r$, we can conclude with

$$(-1)^{2m+2n} n!m!\int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\mathrm{d}x\Big)\mathrm{d}y. \tag{97}$$

By noting that the term

$$(-1)^{2m+2n} = (-1)^{2(m+n)} \tag{98}$$

is positive whatever the value of n and m is, and by the fact that

$$\int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\mathrm{d}x\Big)\mathrm{d}y = \int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{1}{2}\left(ax^2+2abxy+cy^2\right)}\mathrm{d}x\Big)\mathrm{d}y = 2\pi\frac{1}{\sqrt{\Delta}}, \tag{99}$$

we finally obtain

$$(-1)^{2m+2n} n!m!\int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}\mathrm{d}x\Big)\mathrm{d}y = n!m!2\pi\frac{1}{\sqrt{\Delta}}, \tag{100}$$

that is

$$\int_{-\infty}^{+\infty}\mathrm{d}y\int_{-\infty}^{+\infty}\mathrm{d}x\mathrm{H}_{m,n}(x,y)\mathrm{G}_{r,s}(x,y)\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}} = n!m!2\pi\frac{1}{\sqrt{\Delta}}, \tag{101}$$

which proves the theorem. □

In previous sections, we have used the orthogonality property satisfied by the one-index Hermite polynomials of the type $\mathrm{He}_m(x)$ and $\mathrm{He}_m(x,y)$, in order to introduce the Hermite functions of one and two variables $\mathrm{he}_m(x)$ and $\mathrm{he}_m(x,y)$. In the same way we can use the result proved in the above theorem to define functions based on the two-index, two-variable Hermite polynomials $H_{m,n}(x,y)$ and their associate $\mathrm{G}_{m,n}(x,y)$, which can verify the bi-orthogonality property.

Definition III.1. Let the Hermite polynomials $\mathrm{H}_{m,n}(x,y)$ and $\mathrm{G}_{m,n}(x,y)$ we call two-index, two-variable Hermite functions, be the functions defined in the following way

$$\overline{\mathrm{H}}_{m,n}(x,y) = \frac{\sqrt[4]{\Delta}}{2\pi}\frac{1}{\sqrt{m!n!}}\mathrm{H}_{m,n}(x,y)\mathrm{e}^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}, \tag{102}$$

$$\overline{\mathrm{G}}_{m,n}(x,y) = \frac{\sqrt[4]{\Delta}}{2\pi}\frac{1}{\sqrt{m!n!}}\mathrm{G}_{m,n}(x,y)\mathrm{e}^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}. \tag{103}$$

It is evident that the two-index, two-variable Hermite functions are bi-orthogonal and in particular, bi-orthonormal. In fact, by applying the result of Theorem III.1, we have

$$\begin{aligned} &\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}y\overline{\mathrm{H}}_{m,n}(x,y)\,\overline{\mathrm{G}}_{r,s}(x,y) \\ &= \frac{\sqrt{\Delta}}{2\pi}\frac{1}{\sqrt{m!n!}}\frac{1}{\sqrt{r!s!}}\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}y\mathrm{H}_{m,n}(x,y)\mathrm{G}_{r,s}(x,y)\mathrm{e}^{-\frac{1}{2}\underline{z}^t\hat{M}\underline{z}}, \end{aligned} \tag{104}$$

and then,

$$\begin{aligned} &\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}y\overline{\mathrm{H}}_{m,n}(x,y)\,\overline{\mathrm{G}}_{r,s}(x,y) \\ &= \frac{\sqrt{\Delta}}{2\pi}\frac{1}{\sqrt{m!n!}}\frac{1}{\sqrt{r!s!}}m!n!\frac{2\pi}{\sqrt{\Delta}}\delta_{m,r}\delta_{n,s} = \delta_{m,r}\delta_{n,s}. \end{aligned} \tag{105}$$

The Hermite polynomials of type $\mathrm{H}_{m,n}(x,y)$ and the related associated $\mathrm{G}_{m,n}(x,y)$ (see [**8**]) were introduced by operating a dimensional increase on the standard Hermite polynomials $\mathrm{He}_m(x)$ by using a two-dimensional vector index acting on a two-dimensional vector variable. At the same time the structure used to define the Hermite polynomials of the form $\mathrm{H}_{m,n}(x,y)$ is based on a quadratic form and then, on a two-dimensional matrix which is invertible. This last fact has led to explore the possibility to introduce a slightly different polynomials recognized as Hermite-type, so that we have defined the associate two-index, two-variable Hermite polynomials of type $\mathrm{G}_{m,n}(x,y)$. It is evident that many of the properties deduced for these polynomials belonging to the class of the generalized Hermite polynomials are a generalization of the same relations presented and discussed for the ordinary Hermite polynomials $\mathrm{He}_m(x)$, and described for both the $\mathrm{H}_{m,n}(x,y)$ and $\mathrm{G}_{m,n}(x,y)$ Hermite polynomials. In particular, it is known that the following result holds

$$\frac{\partial}{\partial\tau}S_{m,n}(x,y;\tau) = -\frac{1}{2}\left(\partial_x\partial_y\right)\hat{M}^{-1}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}S_{m,n}(x,y;\tau) \tag{106}$$

satisfying the conditions at $\tau = 0$,

$$S_{m,n}(x, y; 0) = \begin{cases} \xi^m \eta^n \\ x^m y^n, \end{cases} \tag{107}$$

respectively.

Since we have defined the two-index, two-variable Hermite functions $\overline{\mathrm{H}}_{m,n}(x, y)$ and $\overline{\mathrm{G}}_{m,n}(x, y)$ by using the related Hermite polynomials $\mathrm{H}_{m,n}(x, y)$ and $\mathrm{G}_{m,n}(x, y)$, we expect to deduce similar relations which involve the above bi-orthogonal Hermite functions and finally to obtain a partial differential equation solved by the Hermite functions of type $\overline{\mathrm{H}}_{m,n}(x, y)$ and $\overline{G}_{m,n}(x, y)$.

Proposition III.1. *The Hermite functions $\overline{H}_{m,n}(x, y)$ satisfy the following recurrence relations*

$$\begin{aligned}&\left[\frac{\partial}{\partial x} + \frac{1}{2}(ax + by)\right]\overline{\mathrm{H}}_{m,n}(x, y) \\ &\qquad = a\sqrt{m}\overline{\mathrm{H}}_{m-1,n}(x, y) + b\sqrt{n}\overline{\mathrm{H}}_{m,n-1}(x, y)\end{aligned} \tag{108}$$

and

$$\begin{aligned}&\left[\frac{\partial}{\partial y} + \frac{1}{2}(bx + cy)\right]\overline{\mathrm{H}}_{m,n}(x, y) \\ &\qquad = b\sqrt{m}\overline{\mathrm{H}}_{m-1,n}(x, y) + c\sqrt{n}\overline{\mathrm{H}}_{m,n-1}(x, y).\end{aligned} \tag{109}$$

Proof. By deriving with respect to x in the definition of the Hermite function $\overline{\mathrm{H}}_{m,n}(x, y)$, we have

$$\frac{\partial}{\partial x}\overline{\mathrm{H}}_{m,n}(x, y) = \frac{\sqrt[4]{\Delta}}{2\pi}\frac{1}{\sqrt{m!n!}}\frac{\partial}{\partial x}\left(\mathrm{H}_{m,n}(x, y)\mathrm{e}^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\right). \tag{110}$$

Let us study the derivative on the r.h.s. of the above equation obtaining

$$\begin{aligned}&\frac{\partial}{\partial x}\left(\mathrm{H}_{m,n}(x, y)\mathrm{e}^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\right) \\ &= \left(\frac{\partial}{\partial x}\mathrm{H}_{m,n}(x, y)\right)\mathrm{e}^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}} + \mathrm{H}_{m,n}(x, y)\frac{\partial}{\partial x}\mathrm{e}^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}.\end{aligned} \tag{111}$$

By applying the recurrence relation

$$\frac{\partial}{\partial x}\mathrm{H}_{m,n}(x, y) = am\,\mathrm{H}_{m-1,n}(x, y) + bn\,\mathrm{H}_{m,n-1}(x, y), \tag{112}$$

we finally have

$$\begin{aligned}&\frac{\partial}{\partial x}\left(\mathrm{H}_{m,n}(x, y)\mathrm{e}^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\right) \\ &= (am\mathrm{H}_{m-1,n}(x, y) + bn\mathrm{H}_{m,n-1}(x, y))\,\mathrm{e}^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}} \\ &\quad - \frac{1}{4}\mathrm{H}_{m,n}(x, y)\left[\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right. \\ &\quad \left. + \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]\mathrm{e}^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}.\end{aligned} \tag{113}$$

Note that the two-index, two-variable Hermite polynomials of type $\mathrm{H}_{m,n}(x,y)$ can be expressed in terms of the Hermite function $\overline{\mathrm{H}}_{m,n}(x,y)$. By making the appropriate manipulations, we end up with

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\mathrm{H}_{m,n}(x,y)\mathrm{e}^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\right)\\
(114)\quad &= am\frac{\sqrt{2\pi}}{\sqrt[4]{\Delta}}\sqrt{(m-1)!n!}\,\overline{\mathrm{H}}_{m-1,n}(x,y) + bn\sqrt{m!(n-1)!}\,\overline{\mathrm{H}}_{m,n-1}(x,y)\\
&\quad -\frac{1}{2}\frac{\sqrt{2\pi}}{\sqrt[4]{\Delta}}\sqrt{m!n!}\,\overline{\mathrm{H}}_{m,n}(x,y)\,(ax+by)
\end{aligned}$$

and then,

$$\begin{aligned}
&\frac{\partial}{\partial x}\overline{\mathrm{H}}_{m,n}(x,y)\\
(115)\quad &= \frac{\sqrt[4]{\Delta}}{\sqrt{2\pi}}\frac{1}{\sqrt{m!n!}}\frac{\sqrt{2\pi}}{\sqrt[4]{\Delta}}\sqrt{m!n!}\\
&\quad\times\left[a\sqrt{m}\,\overline{\mathrm{H}}_{m-1,n}(x,y)+b\sqrt{n}\,\overline{\mathrm{H}}_{m,n-1}(x,y)-\frac{1}{2}\overline{\mathrm{H}}_{m,n}(x,y)\,(ax+by)\right],
\end{aligned}$$

which proves the first recurrence relation in the statement.

To show the second relation, we derive with respect to y again in the definition of the Hermite functions $\overline{\mathrm{H}}_{m,n}(x,y)$

$$(116)\qquad \frac{\partial}{\partial y}\overline{\mathrm{H}}_{m,n}(x,y) = \frac{\sqrt[4]{\Delta}}{\sqrt{2\pi}}\frac{1}{\sqrt{m!n!}}\frac{\partial}{\partial y}\left(\mathrm{H}_{m,n}(x,y)\mathrm{e}^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\right).$$

Following the same procedure used above, we can easily prove the second recurrence relations. □

We have introduced the Hermite functions and their adjoint by using he structure of the Hermite polynomials of type $\mathrm{H}_{m,n}(x,y)$ and $\mathrm{G}_{m,n}(x,y)$. As we have seen in the above statement, it is possible to derive similar relations for these Hermite functions of type $\overline{\mathrm{H}}_{m,n}(x,y)$ by using the techniques and the operational properties of two-index, two-variable Hermite polynomials.

Proposition III.2. *The bi-orthogonal Hermite functions of type $\overline{\mathrm{H}}_{m,n}(x,y)$ verify the following relations*

$$(117)\qquad \begin{aligned}\sqrt{m+1}\,\overline{\mathrm{H}}_{m+1,n}(x,y) &= (ax+by)\overline{\mathrm{H}}_{m,n}(x,y)\\ &\quad - a\sqrt{m}\,\overline{\mathrm{H}}_{m-1,n}(x,y) - b\sqrt{n}\,\overline{\mathrm{H}}_{m,n-1}(x,y)\end{aligned}$$

$$(118)\qquad \begin{aligned}\sqrt{n+1}\,\overline{\mathrm{H}}_{m,n+1}(x,y) &= (bx+cy)\overline{\mathrm{H}}_{m,n}(x,y)\\ &\quad - b\sqrt{m}\,\overline{\mathrm{H}}_{m-1,n}(x,y) - c\sqrt{n}\,\overline{\mathrm{H}}_{m,n-1}(x,y)\end{aligned}$$

Proof. We note that the Hermite function of indexes $m+1$, n, reads

$$(119)\qquad \overline{\mathrm{H}}_{m+1,n}(x,y) = \frac{\sqrt[4]{\Delta}}{\sqrt{2\pi}}\frac{1}{\sqrt{m+1}}\frac{1}{\sqrt{m!n!}}\mathrm{H}_{m+1,n}(x,y)\mathrm{e}^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}},$$

where we can substitute the recurrence relation satisfied by Hermite polynomials of type $\mathrm{H}_{m,n}(x,y)$:

$$\begin{aligned} \mathrm{H}_{m+1,n}(x,y) = {} & (ax+by)\,\mathrm{H}_{m,n}(x,y) \\ & - am\,\mathrm{H}_{m-1,n}(x,y) - bn\,\mathrm{H}_{m,n-1}(x,y) \end{aligned} \tag{120}$$

to obtain

$$\begin{aligned} \sqrt{m+1}\;\overline{\mathrm{H}}_{m+1,n}\,(x,y) = {} & \frac{\sqrt[4]{\Delta}}{\sqrt{2\pi}}\frac{1}{\sqrt{m!n!}}\mathrm{e}^{-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\big[(ax+by)\mathrm{H}_{m,n}(x,y) \\ & - am\mathrm{H}_{m-1,n}(x,y) - bnH_{m,n-1}(x,y)\big]. \end{aligned} \tag{121}$$

By using the definition of Hermite function $\overline{H}_{m,n}\,(x,y)$, the above equation can be write in the form

$$\begin{aligned} & \sqrt{m+1}\;\overline{H}_{m+1,n}\,(x,y) \\ & \quad = \frac{\sqrt[4]{\Delta}}{\sqrt{2\pi}}\frac{1}{\sqrt{m!n!}}\mathrm{e}^{-\frac{1}{4}\underline{z}^t\,\hat{M}\underline{z}}\Big[(ax+by)\mathrm{e}^{\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\sqrt{m!n!}\;\overline{H}_{m,n}\,(x,y) \\ & \quad\quad - am\frac{\sqrt{2\pi}}{\sqrt[4]{\Delta}}\sqrt{(m-1)!n!}\mathrm{e}^{\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\;\overline{H}_{m-1,n}\,(x,y) \\ & \quad\quad - bn\frac{\sqrt{2\pi}}{\sqrt[4]{\Delta}}\sqrt{m!(n-1)!}\mathrm{e}^{\frac{1}{4}\underline{z}^t\hat{M}\underline{z}}\;\overline{H}_{m,n-1}\,(x,y)\Big], \end{aligned} \tag{122}$$

that is, once recast, the first expression of the present proposition.

In the analogous way it is possible to prove the second recurrence relation by using again the recurrence relation related to the Hermite polynomials of type $\mathrm{H}_{m,n}(x,y)$. □

The relations derived in the above propositions can be used to define useful operators acting on the Hermite functions of type $\overline{\mathrm{H}}_{m,n}\,(x,y)$.

By manipulating the first of recurrence relations present in Proposition III.2, we have

$$\begin{aligned} a\sqrt{m}\;\overline{H}_{m-1,n}\,(x,y) = {} & (ax+by)\overline{H}_{m,n}(x,y) \\ & - b\sqrt{n}\;\overline{H}_{m,n-1}(x,y) - \sqrt{m+1}\;\overline{H}_{m+1,n}(x,y) \end{aligned} \tag{123}$$

which, once substitute in the first equation of Proposition III.2, gives

$$\Big[-\frac{\partial}{\partial x} + \frac{1}{2}(ax+by)\Big]\overline{\mathrm{H}}_{m,n}(x,y) = \sqrt{m+1}\;\overline{\mathrm{H}}_{m+1,n}(x,y). \tag{124}$$

In the same way by using the second expressions stated in Proposition III.2 and Proposition III.1, we obtain

$$\Big[\frac{1}{2}(bx+cy) - \frac{\partial}{\partial y}\Big]\overline{\mathrm{H}}_{m,n}(x,y) = \sqrt{n+1}\;\overline{\mathrm{H}}_{m,n+1}(x,y). \tag{125}$$

The recurrence relations stated in the previous two propositions can be also used to derive further differential expressions regarding the bi-orthogonal Hermite functions. In fact, following the same procedure used above, in particular, alternately combining the first and the second expression of Proposition III.2 with the second and the first of Proposition III.1, it is possible to complete the characterization with regard to the differential properties satisfied by the Hermite functions of type $\overline{\mathrm{H}}_{m,n}(x,y)$. Indeed, we have the following relations

$$\left[-\frac{1}{\Delta}\left(b\frac{\partial}{\partial x}-a\frac{\partial}{\partial y}\right)+\frac{1}{2}y\right]\overline{\mathrm{H}}_{m,n}(x,y)=\sqrt{n}\,\overline{\mathrm{H}}_{m,n-1}(x,y) \tag{126}$$

$$\left[-\frac{1}{\Delta}\left(c\frac{\partial}{\partial x}-b\frac{\partial}{\partial y}\right)+\frac{1}{2}x\right]\overline{\mathrm{H}}_{m,n}(x,y)=\sqrt{m}\,\overline{\mathrm{H}}_{m-1,n}(x,y) \tag{127}$$

It is an evident analogy between the four relations presented above and the expressions regarding the two-index, two-variable Hermite polynomials of type $\mathrm{H}_{m,n}(x,y)$.

Definition III.2. Given the Hermite functions $\overline{\mathrm{H}}_{m,n}(x,y)$, we define the related shift operators by setting

$$\hat{a}_{+,0}=\frac{1}{2}(ax+by)-\frac{\partial}{\partial x}, \tag{128}$$

$$\hat{a}_{0,+}=\frac{1}{2}(bx+cy)-\frac{\partial}{\partial y}, \tag{129}$$

and

$$\hat{a}_{-,0}=\frac{1}{\Delta}\left(c\frac{\partial}{\partial x}-b\frac{\partial}{\partial y}\right)+\frac{1}{2}x, \tag{130}$$

$$\hat{a}_{0,-}=-\frac{1}{\Delta}\left(b\frac{\partial}{\partial x}-a\frac{\partial}{\partial y}\right)+\frac{1}{2}y. \tag{131}$$

The above operators are free from any parameters, not presenting any index variable in their structure; therefore, different from the shift operators related to Hermite polynomials of type $\mathrm{H}_{m,n}(x,y)$.

It could be useful to summarize the action of these operators

$$\begin{cases}\hat{a}_{+,0}\overline{\mathrm{H}}_{m,n}(x,y)=\sqrt{m+1}\overline{\mathrm{H}}_{m+1,n}(x,y),\\ \hat{a}_{0,+}\overline{\mathrm{H}}_{m,n}(x,y)=\sqrt{n+1}\overline{\mathrm{H}}_{m,n+1}(x,y),\\ \hat{a}_{-,0}\overline{\mathrm{H}}_{m,n}(x,y)=\sqrt{m}\overline{\mathrm{H}}_{m-1,n}(x,y),\\ \hat{a}_{0,-}\overline{\mathrm{H}}_{m,n}(x,y)=\sqrt{n}\overline{\mathrm{H}}_{m,n-1}(x,y).\end{cases} \tag{132}$$

As mentioned above and by virtue of the relations established above, we can proceed to state the important result concerning the partial differential equation solved by the bi-orthogonal Hermite functions $\overline{\mathrm{H}}_{m,n}(x,y)$ and $\overline{\mathrm{G}}_{m,n}(x,y)$. We proceed by presenting the results for the Hermite functions of type $\overline{\mathrm{H}}_{m,n}(x,y)$ and later we discuss the case for the related associated Hermite functions.

Theorem III.2. *The bi-orthogonal Hermite functions solve the following partial differential equation*

$$\left[-\underline{\partial}_z^t\hat{M}^{-1}\underline{\partial}_z-\left(m+n+1-\frac{1}{4}\underline{z}^t\hat{M}\underline{z}\right)\right]\overline{\mathrm{H}}_{m,n}(x,y)=0, \tag{133}$$

where

$$\underline{\partial}_z = \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix}. \tag{134}$$

Proof. We consider the following operational relations deriving from the above considerations

$$\hat{a}_{+,0}\left[\hat{a}_{-,0}\overline{\mathrm{H}}_{m,n}(x,y)\right] = m\overline{\mathrm{H}}_{m,n}(x,y), \tag{135}$$

$$\hat{a}_{0,+}\left[\hat{a}_{0,-}\overline{\mathrm{H}}_{m,n}(x,y)\right] = n\overline{\mathrm{H}}_{m,n}(x,y). \tag{136}$$

which can be explicited to obtain

$$\left[\frac{1}{2}(ax+by) - \frac{\partial}{\partial x}\right]\left[\frac{1}{\Delta}\Big(c\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\Big) + \frac{1}{2}x\right]\overline{\mathrm{H}}_{m,n}(x,y) = m\overline{\mathrm{H}}_{m,n}(x,y), \tag{137}$$

$$\left[\frac{1}{2}(bx+cy) - \frac{\partial}{\partial y}\right]\left[-\frac{1}{\Delta}\Big(b\frac{\partial}{\partial x} - a\frac{\partial}{\partial y}\Big) + \frac{1}{2}y\right]\overline{\mathrm{H}}_{m,n}(x,y) = n\overline{\mathrm{H}}_{m,n}(x,y). \tag{138}$$

The operator in the first of the above relations can be recast in the form

$$\begin{aligned}&\frac{1}{2\Delta}\Big[c(ax+by)\frac{\partial}{\partial x}\Big] - \frac{1}{2\Delta}\Big[b(ax+by)\frac{\partial}{\partial y}\Big] + \frac{1}{4}\left(ax^2+bxy\right)\\ &\qquad + \frac{1}{\Delta}\Big(b\frac{\partial^2}{\partial x\partial y} - c\frac{\partial^2}{\partial x^2}\Big) - \frac{1}{2} - \frac{1}{2}x\frac{\partial}{\partial x}\end{aligned} \tag{139}$$

and regarding the second equation, we can rewrite the operator as follows:

$$\begin{aligned}&-\frac{1}{2\Delta}\Big[b(bx+cy)\frac{\partial}{\partial x} - a(bx+cy)\frac{\partial}{\partial y}\Big] + \frac{1}{4}y\,(bx+cy)\\ &\qquad + \frac{1}{\Delta}\Big(b\frac{\partial^2}{\partial x\partial y} - a\frac{\partial^2}{\partial y^2}\Big) - \frac{1}{2} - \frac{1}{2}y\frac{\partial}{\partial y}.\end{aligned} \tag{140}$$

After substituting the above expressions in the operational relations and making a sum of these relations member to member, we obtain

$$\begin{aligned}&\Big\{\frac{1}{2\Delta}\Big[c(ax+by)\frac{\partial}{\partial x}\Big] - \frac{1}{2\Delta}\Big[b(ax+by)\frac{\partial}{\partial y}\Big] + \frac{1}{4}(ax^2+bxy)\\ &\quad + \frac{1}{\Delta}\Big(b\frac{\partial^2}{\partial x\partial y} - c\frac{\partial^2}{\partial x^2}\Big) - \frac{1}{2} - \frac{1}{2}x\frac{\partial}{\partial x}\\ &\quad - \frac{1}{2\Delta}\Big[b(bx+cy)\frac{\partial}{\partial x} - a(bx+cy)\frac{\partial}{\partial y}\Big]\\ &\quad + \frac{1}{4}y\,(bx+cy) + \frac{1}{\Delta}\Big(b\frac{\partial^2}{\partial x\partial y} - a\frac{\partial^2}{\partial y^2}\Big) - \frac{1}{2} - \frac{1}{2}y\frac{\partial}{\partial y}\Big\}\overline{\mathrm{H}}_{m,n}(x,y)\\ &= (m+n)\overline{\mathrm{H}}_{m,n}(x,y).\end{aligned} \tag{141}$$

We note that, by using the definition of two-index, two-variable Hermite polynomials $\mathrm{H}_{m,n}(x,y)$, the following relations hold

$$\frac{1}{4}\left(ax^2+2bxy+cy^2\right) = \frac{1}{4}\underline{z}^t\hat{M}\underline{z}, \tag{142}$$

$$-\underline{\partial}_z^t \hat{M}^{-1}\underline{\partial}_z = -\frac{1}{\Delta}\Big(c\frac{\partial^2}{\partial x^2} - 2b\frac{\partial^2}{\partial x\partial y} + a\frac{\partial^2}{\partial y^2}\Big) \tag{143}$$

and then, we can recast the operator in the l.h.s of equation (141) in the form

$$\Big[-\underline{\partial}_z^t \hat{M}^{-1}\underline{\partial}_z + \frac{1}{4}\underline{z}^t \hat{M}\underline{z} - 1\Big]\overline{\mathrm{H}}_{m,n}(x,y) = (m+n)\overline{\mathrm{H}}_{m,n}(x,y), \tag{144}$$

which easily gives the statement of the theorem. □

IV. The adjoint bi-orthogonality of Hermite functions

We can now establish analogous results for the adjoint bi-orthogonal Hermite functions of type $\overline{\mathrm{G}}_{m,n}(x,y)$. By considering the link that exists between the two-index, two-variable Hermite polynomials and their adjoint and moreover, between the present Hermite functions and the related associated functions, we proceed in a non-repetitive way acting directly on the operators presented in Definition III.2, we consider the following vectorial operator

$$\hat{a}^+ = \begin{pmatrix} \hat{a}_{+,0} \\ \hat{a}_{0,+} \end{pmatrix}. \tag{145}$$

We can easily prove that

$$\hat{a}^+ = \frac{1}{2}\hat{M}\underline{z} - \underline{\partial}_z. \tag{146}$$

In fact, the r.h.s. in the above equation can be expressed

$$\begin{aligned} \frac{1}{2}\hat{M}\underline{z} - \underline{\partial}_z &= \frac{1}{2}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} ax+by \\ bx+cy \end{pmatrix} - \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} \end{aligned} \tag{147}$$

and then by the first relation in Definition III.2, we find

$$\begin{aligned} \frac{1}{2}(ax+by) - \frac{\partial}{\partial x} &= \hat{a}_{+,0}, \\ \frac{1}{2}(bx+cy) - \frac{\partial}{\partial y} &= \hat{a}_{0,+}, \end{aligned} \tag{148}$$

which proves the statement. In the same way by setting

$$\hat{a}^- = \begin{pmatrix} \hat{a}_{-,0} \\ \hat{a}_{0,-} \end{pmatrix}, \tag{149}$$

we obtain the further relation

$$\hat{a}^- = \hat{M}^{-1}\underline{\partial}_z + \frac{1}{2}\underline{z}. \tag{150}$$

Now we use the two-vector operators defined above for the Hermite functions of type $\overline{\mathrm{H}}_{m,n}(x,y)$ to determine the corresponding creation and annihilation operators for the associated Hermite functions $\overline{\mathrm{G}}_{m,n}(x,y)$. We know that the structural difference between the two-index, two-variable Hermite polynomials and their associated is essentially different in the matrix of their quadratic form that defines them. Otherwise, the Hermite functions are defined by using the Hermite polynomials of type $\mathrm{H}_{m,n}(x,y)$ and $\mathrm{G}_{m,n}(x,y)$. This aspect leads us to define the creation and annihilation operators for the bi-orthogonal Hermite functions of type $\overline{G}_{m,n}(x,y)$ by modifying the corresponding operators obtained for the Hermite functions $\overline{\mathrm{H}}_{m,n}(x,y)$ directly.

We remind that the adjoint quadratic form of the two-index, two-variable Hermite polynomials of type $\mathrm{H}_{m,n}(x,y)$, is expressed by

$$\overline{q}\left(\underline{z}\right) = \underline{z}^t \hat{M}^{-1} \underline{z} \tag{151}$$

which introduces the vectorial variable $\underline{v} = \hat{M}\underline{z}$, where $\underline{v} = \binom{\xi}{\eta}$, to define the associated Hermite polynomials of the form $\mathrm{G}_{m,n}(x,y)$. By using the above relations, we introduce the operators regarding the associated Hermite functions $\overline{\mathrm{G}}_{m,n}(x,y)$ by setting

$$\hat{B}^{+} = \frac{1}{2}\hat{M}^{-1}\underline{v} - \underline{\partial}_v \tag{152}$$

and

$$\hat{B}^{-} = \hat{M}\underline{\partial}_v + \frac{1}{2}\underline{v}. \tag{153}$$

It is evident that the above expressions refer to the vectorial variable $\underline{v}$ and then, we need to express the creation and annihilation operators related to the associated Hermite functions $\overline{\mathrm{G}}_{m,n}(x,y)$ in terms of the vectorial variable $\underline{z}$. By using the link between the variables $\underline{z}$ and $\underline{v}$, we immediately get

$$\frac{1}{2}\hat{M}^{-1}\underline{v} - \underline{\partial}_v = \frac{1}{2}\underline{z} - \hat{M}^{-1}\underline{\partial}_z, \tag{154}$$

$$\hat{M}\underline{\partial}_v + \frac{1}{2}\underline{v} = \underline{\partial}_z + \frac{1}{2}\hat{M}\underline{z} \tag{155}$$

and then, we can rewrite the creation and annihilation operators in the following form

$$\hat{B}^{+} = \frac{1}{2}\underline{z} - \hat{M}^{-1}\underline{\partial}_z, \tag{156}$$

$$\hat{B}^{-} = \underline{\partial}_z + \frac{1}{2}\hat{M}\underline{z}. \tag{157}$$

It is now possible to obtain an explicit form of the creation and annihilation operators related to the associated Hermite functions $\overline{\mathrm{G}}_{m,n}(x,y)$. From the first expression, we have

$$\hat{B}^{+} = \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{\Delta}\begin{pmatrix} -c & b \\ b & -a \end{pmatrix}\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} = \begin{cases} \frac{1}{2}x - \frac{1}{\Delta}\left(c\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right) \\ \frac{1}{2}y - \frac{1}{\Delta}\left(-b\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}\right) \end{cases} \tag{158}$$

and in analogous way for the second operator, we get

$$\text{(159)}\quad \hat{B}^{-} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \dfrac{\partial}{\partial x} + \dfrac{1}{2}(ax+by) \\ \dfrac{\partial}{\partial y} + \dfrac{1}{2}(bx+cy) \end{cases}$$

Finally we can state the explicit form for the creation and annihilation operators related to the Hermite functions $\overline{\mathrm{G}}_{m,n}(x,y)$. For the creation operators, we obtain

$$\text{(160)}\quad \hat{B}_{+,0} = \frac{1}{2}x - \frac{1}{\Delta}\left(c\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right), \qquad \text{and} \qquad \hat{B}_{0,+} = \frac{1}{2}y - \frac{1}{\Delta}\left(-b\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}\right)$$

and similarly, for the annihilation operators, we get

$$\text{(161)}\quad \hat{B}_{-,0} = \frac{\partial}{\partial x} + \frac{1}{2}(ax+by), \qquad \text{and} \qquad \hat{B}_{0,-} = \frac{\partial}{\partial y} + \frac{1}{2}(bx+cy).$$

We have defined the above operators by using the concepts and the related formalism of the creation and annihilation operators introduced for the Hermite bi-orthogonal functions of type $\overline{\mathrm{H}}_{m,n}(x,y)$. We expect that these operators have the same effect on the associated Hermite functions of the form $\overline{\mathrm{G}}_{m,n}(x,y)$. In fact, we immediately obtain the fundamental relations

$$\text{(162)}\quad \begin{cases} \hat{B}_{+,0}\,\overline{\mathrm{G}}_{m,n}(x,y) = \sqrt{m+1}\colon \overline{\mathrm{G}}_{m+1,n}(x,y), \\ \hat{B}_{0,+}\,\overline{\mathrm{G}}_{m,n}(x,y) = \sqrt{n+1}\colon \overline{\mathrm{G}}_{m,n+1}(x,y), \\ \hat{B}_{-,0}\,\overline{\mathrm{G}}_{m,n}(x,y) = \sqrt{m}\colon \overline{\mathrm{G}}_{m-1,n}(x,y), \\ \hat{B}_{0,-}\,\overline{\mathrm{G}}_{m,n}(x,y) = \sqrt{n}\colon \overline{\mathrm{G}}_{m,n-1}(x,y), \end{cases}$$

which conclude that the operators related to the adjoint function $\overline{\mathrm{G}}_{m,n}(x,y)$ act in symmetric ways as the operators introduced in Definition III.2 for the functions $\overline{\mathrm{H}}_{m,n}(x,y)$.

The operational techniques showed in this paper could be generalized for many families of polynomials, for example, the Laguerre or Legendre polynomials (see [**11, 12**]). The property of bi-orthogonality can be used to derive some interesting results in the field of harmonic oscillator (see [**13, 14**]) and their represent an important tool to investigate the properties of annihilation and creation operators. The techniques showed in the paper represent a powerful tool to investigate the properties of many families of special functions, as for example generalized Bessel functions (see [**15**]).

The above presented polynomials can also be used for more refined numerical fitting techniques and their applications in electromagnetic problems (see [**16, 17, 18, 19**]).

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C. Cesarano, Università Telematica Internazionale Uninettuno, C.so Vittorio Emanuele II, 39, Rome, Italy, *e-mail*: `c.cesarano@uninettunouniversity.net`

C. Fornaro, Università Telematica Internazionale Uninettuno, C.so Vittorio Emanuele II, 39, Rome, Italy, *e-mail*: `c.fornaro@uninettunouniversity.net`

L. Vazquez, Departamento de Matematica Aplicada, Facultad de Informatica, Universidad Complutense de Madrid, Ciudad Universitaria 28040, Madrid, Spain, *e-mail*: `lvazquez@fdi.ucm.es`