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EXPLORING PROJECTIVE NORM GRAPHS

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ABSTRACT. The projective norm graphs $\mathrm{NG}(q,t)$ provide tight constructions for the Turán number of complete bipartite graphs $K_{t,s}$ with $s > (t-1)!$. The determination of the largest integer s_t, such that the projective norm graph $\mathrm{NG}(q,t)$ contains K_{t,s_t} for all large enough prime powers q is an important open question with far-reaching general consequences. Here we settle the case $t = 4$. Along the way we also develop methods to count the copies of any fixed 3-degenerate subgraph, and find that projective norm graphs are quasirandom with respect to this parameter. Some of these results also extend the work of Alon and Shikhelman on generalized Turán numbers. Finally we also completely determine the automorphism group of $\mathrm{NG}(q,t)$ for every possible values of the parameters.

1. Introduction

Given a graph H and integer $n \in \mathbb{N}$, the Turán number of H, denoted by $\mathrm{ex}(n,H)$, is the maximum number of edges a simple H-free graph on n vertices may have. For general H, as a corollary of the Erdős-Stone Theorem, Erdős and Simonovits proved that $\mathrm{ex}(n,H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2)$, where $\chi(H)$ is the chromatic number of H. If H is not bipartite, this theorem determines $\mathrm{ex}(n,H)$ asymptotically, however for bipartite graphs it merely states that $\mathrm{ex}(n,H)$ is of lower than quadratic order. A general classification of the order of magnitude of bipartite Turán numbers is widely open, even in the simplest-looking cases of even cycles and complete bipartite graphs [**6**]. For even cycles the order of magnitude of $\mathrm{ex}(n,C_k)$ is known only for $k = 4,6,10$. For complete bipartite graphs a general upper bound $\mathrm{ex}(n,K_{t,s}) \le \frac{1}{2}\sqrt[t]{s-1}\cdot n^{2-\frac{1}{t}} + \frac{t-1}{2}\cdot n$ was proved by Kővári, T. Sós and Turán using an elementary double counting argument. In general it is commonly conjectured that the order of magnitude in the Kővári-T.Sós-Turán theorem is the right one.

Conjecture 1. For every $t,s \in \mathbb{N}$, $t \le s$, $\mathrm{ex}(n,K_{t,s}) = \Theta\left(n^{2-\frac{1}{t}}\right)$.

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A general lower bound of $\Omega(n^{2-\frac{s+t-2}{st-1}})$ can be obtained using the probabilistic method, but this is of smaller order for all values of the parameters. Constructions matching the order of the upper bound were first found for $K_{2,2}$-free graphs by Klein and later for $K_{3,3}$-free graphs by Brown. In both cases further analysis has also led to the determination of the correct leading coefficient (see e.g. [**6**]).

Alon, Rónyai and Szabó [**2**], by modifying a construction of Kollár, Rónyai and Szabó [**8**], proved Conjecture 1 for $t \geq 2$, $s > (t-1)!$ by constructing a family of graphs, called projective norm graphs, that are $K_{t,(t-1)!+1}$-free and their density matches the order of magnitude of the Kővári-Sós-Turán upper bound.

2. The projective norm graphs

Let q be a prime power, $t \geq 2$ a positive integer and let $N\colon \mathbb{F}_{q^{t-1}} \to \mathbb{F}_q$ denote the $\mathbb{F}_q$-norm on $\mathbb{F}_{q^{t-1}}$, i.e. $N(A) = A \cdot A^q \cdot A^{q^2} \cdots A^{q^{t-2}}$ for $A \in \mathbb{F}_{q^{t-1}}$. Then the projective norm graph $\mathrm{NG}(q,t)$ has vertex set $\mathbb{F}_{q^{t-1}} \times \mathbb{F}_q^*$ and two vertices (A,a) and (B,b) are adjacent if and only if $N(A+B) = ab$[1]. Clearly, $\mathrm{NG}(q,t)$ has $n = n(\mathrm{NG}(q,t)) = q^{t-1}\cdot(q-1) = (1+o(1))q^t$ vertices and it is a straightforward calculation to check that the number of edges is $e = e(\mathrm{NG}(q,t)) = (1+o(1))\frac{1}{2}q^{2t-1} = (1+o(1))\frac{1}{2}n^{2-\frac{1}{t}}$. Using a general algebro-geometric lemma [**8**], it was shown [**2**] that $\mathrm{NG}(q,t)$ is $K_{t,(t-1)!+1}$-free and since it also has the desired density, it verifies Conjecture 1 for $s > (t-1)!$. Since their first appearance, projective norm graphs served as important examples in many other areas of mathematics as well.

A drawback of the proof of the $K_{t,(t-1)!+1}$-freeness of $\mathrm{NG}(q,t)$ is that it does not give any information about complete bipartite subgraphs with any other parameters. In particular, it is not even known whether $\mathrm{NG}(q,t)$ contains a $K_{t,(t-1)!}$. Considering the fundamental nature of Conjecture 1, it was already suggested in [**2**] that the determination of the largest integer s_t, such that $\mathrm{NG}(q,t)$ contains K_{t,s_t} for every large enough prime power q is a question of great interest. It is rather easy to see that $s_2 = 1$ and $s_3 = 2$, but the general bounds for $t \geq 4$ are very far apart: $t-1 \leq s_t \leq (t-1)!$. If s_t were found to be less than $(t-1)!$ then the projective norm graphs verified Conjecture 1 for more values of the parameters than what is known currently. The generality of the key lemma used in [**2**] gives reason for some optimism here.

Recently Grosu showed that there is a sequence of primes of density $\frac{1}{9}$, such that for any prime p in this sequence $\mathrm{NG}(p,4)$ does contain a $K_{4,6}$. Here, as our first main result, we greatly extend this result.

Theorem 1. *$NG(q,4)$ does contain a copy of $K_{4,6}$ for any prime power $q = p^k \geq 5$. In particular we have $s_4 = 6$.*

We remark that for $q = 2$ it is immediate that it does not contain a $K_{4,6}$, and for $q \in \{3,4\}$ one can easily check the same with a computer.

[1] For technical reasons here we allow the two vertices to be the same, i.e. we allow loop edges.

3. Common neighbourhoods

The proof of Theorem 1 is based on a detailed analysis of the common neighbourhoods of small sets of vertices.

For a set of vertices $T \subseteq V(\mathrm{NG}(q,t))$ let us denote by $\deg(T)$ the size of the common neighbourhood of the vertices in T. We call a set of vertices in $\mathrm{NG}(q,t)$ *generic*, if the first coordinates of them are pairwise distinct. In particular, the common neighborhood of non-generic vertex sets is empty. Also, a set of vertices we be referred to as *aligned* if all its elements have the same second coordinate and for $T \subseteq V$ we set $\xi(T) = 1$ if T is aligned and $\xi(T) = 0$ otherwise. Furthermore, for q odd let $\eta_{\mathbb{F}_q}$ be the quadratic character of $\mathbb{F}_q$. With all this notation in hand we can now state our second main result about the sizes of common neighbourhoods of small sets of vertices.

Theorem 2. *Let $q = p^k$ be a prime power, $t \geq 2$ an integer, and consider a generic j-subset $T = \{(A_i, a_i) : i = 1, \ldots, j\}$ of vertices in $NG(q,t)$.*
(a) *If $|T| = 2$, then $\deg(T) = \frac{q^{t-1}-1}{q-1} - \xi(T)$.*
(b) *If $|T| = 3$ and q is odd, then*

$$\deg(T) = \begin{cases} 1 - \eta_{\mathbb{F}_q}\left((1 + c_1 - c_2)^2 - 4c_1\right) - \xi(T) & \text{if } t = 3, \\ 2q + 1 - \eta_{\mathbb{F}_q}(-3) - \xi(T) & \text{if } t = 4,\ (c_1, c_2) = (1, -1), \\ q^{t-3} + O(q^{t-3.5}) & \text{otherwise}, \end{cases}$$

where $c_1 = c_1(T) = \frac{a_1}{a_3} \cdot N\left(\frac{A_2 - A_3}{A_1 - A_2}\right) \in \mathbb{F}_q$, $c_2 = c_2(T) = \frac{a_2}{a_3} \cdot N\left(\frac{A_1 - A_3}{A_1 - A_2}\right) \in \mathbb{F}_q$.
(c) *If $|T| = 4$ and $t \geq 4$ then $\deg(T) \leq 6(q^{t-4} + q^{t-5} + \cdots + q + 1)$.*

One interesting feature of part (*c*) is that its proof provides a new, more elementary argument for the $K_{4,7}$-freeness of $\mathrm{NG}(q,4)$.

4. Quasirandomness

Szabó [**9**] (and independently Alon and Rödl [**1**]) showed that projective norm graphs are quasirandom. This means that to some extent they behave like random graphs. It is an interesting problem to determine to what extent does this random behavior hold. There are definitely limits, as for example the Erdős-Rényi random graph on the same number of vertices and with the same edge density is expected to contain a $K_{t,(t-1)!+1}$, while $\mathrm{NG}(q,t)$ is $K_{t,(t-1)!+1}$ -free. For two graphs G and H let us denote by $X_H(G)$ the number of labaled copies of H in G. We say that a (sequence of) graph(s) $G = G(n)$ on n vertices with edge density $p = p(n)$ is *H-quasirandom* if it contains roughly the same number of labeled copies of H as the Erdős-Rényi random graph $G(n,p)$ is expected to contain, i.e. $X_H(G) = \Theta\left(n^{vH} p^{e(H)}\right)$. If we actually have $X_H(G) = (1 + o(1))\left(n^{vH} p^{e(H)}\right)$ then we say that G is *asymptotically H-quasirandom*.

Theorem 2 allows us to estimate $X_H(\mathrm{NG}(q,t))$ for many graphs H and so to explore the quasirandomness of $\mathrm{NG}(q,t)$.

Theorem 3. *Let $q = p^k$ be an odd prime power and H a simple graph. If H is 3-degenerate and $t \geq 4$ then $NG(q,t)$ is H-quasirandom. Moreover, if H is 3-degenerate and $t \geq 5$ or H is 2-degenerate and $t \geq 3$, then $NG(q,t)$ is asymptotically H-quasirandom.*

As $\mathrm{NG}(q,2)$ does not contain $K_{2,2}$ and $\mathrm{NG}(q,3)$ does not contain $K_{3,3}$, the bound on t in the first part is best possible for both 3- and 2-degenerate graphs. We conjecture though that the stronger statement in the second part should also be true for 3-degenerate graphs and $t = 4$. We also remark that the theorem remains valid even if $H = H_q$ and $v = v(H_q)$ grows moderately, namely if $v(H_q)) = o(\sqrt{q})$ as q tends to infinity, with an error term $o\left(q^{tv(H)-e(H)}\right)$ in the second part.

For graphs H with $\Delta = \Delta(H) \leq \frac{t}{2}$ already the Expander Mixing Lemma implies that $\mathrm{NG}(q,t)$ is H-quasirandom. For $\Delta = 2$ this statement starts to work when t is at least 4 and for $\Delta = 3$ when t is at least 6. Theorem 3 goes beyond the Expander Mixing Lemma and improves the bound on t when $\Delta \leq 3$, in particular, it implies that $\mathrm{NG}(q,4)$ is K_4-quasirandom. Furthermore, it also deals with the much wider class of degenerate graphs rather then merely bounded degree graphs.

5. Generalized Turán numbers

For two simple graphs T and H (with no isolated vertices) and a positive integer n the generalized Turán problem asks for the maximum possible number $\mathrm{ex}(n,T,H)$ of unlabeled copies of T in an H-free graph on n vertices. Note that by setting $T = K_2$ we recover the original Turán problem for H. A systematic study of this function was done recently by Alon and Shikhelman [**3**]. Among others, they have shown that for $s > (t-1)!$ we have

$$\mathrm{ex}(n,T,K_{t,s}) = \Theta\left(n^{v(T)-\frac{e(T)}{t}}\right),$$

whenever $T = K_m$ with $m \leq \frac{t+2}{2}$ or $T = K_{a,b}$ with $a \leq b \leq \frac{t}{2}$. Using Theorem 3 we managed to extend the validity of their result.

Theorem 4. *For every $t \geq 4$ and $s > (t-1)!$ we have*

$$\mathrm{ex}(n,T,K_{t,s}) = \Theta\left(n^{v(T)-\frac{e(T)}{t}}\right),$$

whenever $T = K_4$ or $T = K_{a,b}$ with $\min\{a,b\} \leq 3$.

6. The automorphism group

Finally, we were also able to determine the automorphism group of $\mathrm{NG}(q,t)$ for every value of the parameters. Below Z_n denotes the cyclic group of order n.

Theorem 5. *For any odd prime power $q = p^k$ and integer $t \geq 2$, the maps*

$$(X,x) \mapsto (C^2 \cdot X^{p^i}, \pm N(C) \cdot x^{p^i})$$

are automorphisms of $NG(q,t)$ *for any* $C \in \mathbb{F}^*_{q^{t-1}}$ *and* $i \in [k(t-1)]$. *For any* $q = 2^k$ *and integer* $t \geq 2$, *the maps*

$$(X,x) \mapsto (C^2 \cdot X^{p^i} + A, N(C) \cdot x^{p^i})$$

are automorphisms of $NG(q,t)$ *for any choice of* $C \in \mathbb{F}^*_{q^{t-1}}$, $A \in \mathbb{F}_{q^{t-1}}$, *and* $i \in [k(t-1)]$. *Moreover, for* $q > 2$ *and* $t \geq 2$ *these include all automorphisms and the automorphism group has the following structural description.*

$$\mathrm{Aut}(NG(q,t)) \simeq \begin{cases} Z_{q^{t-1}-1} \rtimes Z_{k(t-1)} & \text{if } q, t-1 \text{ are both odd} \\ \left(Z_2 \times Z_{\frac{q^{t-1}-1}{2}}\right) \rtimes Z_{k(t-1)} & \text{if } q \text{ is odd, } t-1 \text{ is even} \\ \left((Z_p)^{k(t-1)} \rtimes Z_{q^{t-1}-1}\right) \rtimes Z_{k(t-1)} & \text{if } q \text{ is even} \end{cases}$$

Note that if $q = 2$ then $\mathrm{NG}(2,t)$ is a complete graph on 2^{t-1} vertices, and so $\mathrm{Aut}(\mathrm{NG}(2,t))$ is the whole symmetric group of order 2^{t-1}. The automorphisms in Theorem 5 are not that difficult to find, the main challenge is to show that their list is complete.

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