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RESILIENCE WITH RESPECT TO HAMILTONICITY IN RANDOM GRAPHS

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Abstract. The local resilience of a graph G with respect to a property $\mathcal{P}$ measures how much one has to change G locally in order to destroy $\mathcal{P}$. We prove 'resilience' versions of several classical results about Hamiltonicity for the graph models $G_{n,p}$ and $G_{n,d}$.

In the setting of random regular graphs, we prove a resilience version of Dirac's theorem. More precisely, we show that, whenever d is sufficiently large compared to $\varepsilon > 0$, a.a.s. the following holds. Let G' be any subgraph of the random n-vertex d-regular graph $G_{n,d}$ with minimum degree at least $(1/2 + \varepsilon)d$. Then G' is Hamiltonian. This proves a conjecture of Ben-Shimon, Krivelevich and Sudakov.

For the binomial random graph $G_{n,p}$, we prove a resilience version of Pósa's Hamiltonicity condition, and show that a natural guess for a resilience version of Chvátal's theorem fails to be true.

1. Introduction

The study of Hamiltonicity has been at the core of graph theory for the past few decades. A graph G is said to be *Hamiltonian* if it contains a cycle which covers all the vertices of G, and this is called a *Hamilton cycle*. It is well-known that the problem of determining whether a graph is Hamiltonian is NP-complete, and thus most results about Hamiltonicity deal with sufficient conditions which guarantee this property, particularly in the form of degree conditions.

1.1. Degree conditions for Hamiltonicity

The most well-known degree condition for Hamiltonicity is due to Dirac, who proved that every graph G on $n \geq 3$ vertices with minimum degree at least $n/2$ is Hamiltonian. Pósa strengthened this result by proving that a graph G with degree sequence $d_1 \leq \cdots \leq d_n$ such that $d_i \geq i + 1$ for all $i < n/2$ is Hamiltonian. This is best possible in the sense that the condition $d_i \geq i + 1$ cannot be reduced for any i. Chvátal generalised this further by essentially characterising all degree sequences which guarantee Hamiltonicity: a graph with degree sequence $d_1 \leq \cdots \leq d_n$ is Hamiltonian if for all $i < n/2$ we have $d_i \geq i + 1$ or $d_{n-i} \geq n - i$.

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1.2. Hamilton cycles in random graphs

The search for Hamilton cycles in various models of random graphs has also been a driving force in the development of this theory. The traditional binomial model $G_{n,p}$, in which each possible edge is added to an n-vertex graph with probability p independently of the other edges, has seen many results in this direction. Pósa proved that $G_{n,p}$ with $p \gg \log n/n$ is a.a.s. Hamiltonian, and Komlós and Szemerédi determined the exact threshold for p. Furthermore, these results can be strengthened to obtain the following hitting time result. Consider a random graph process as follows: given a set of n vertices, add each of the $\binom{n}{2}$ possible edges, one by one, by choosing the next edge uniformly at random among those that have not been added yet. In this setting, Ajtai, Komlós and Szemerédi [**1**] and Bollobás [**4**] independently proved that a.a.s. the resulting graph becomes Hamiltonian as soon as its minimum degree is at least 2.

The search for Hamilton cycles in other random graph models has proven more difficult. Here we will deal with random *regular* graphs: given $n, d \in \mathbb{N}$ such that $d < n$ and nd is even, $G_{n,d}$ is chosen uniformly at random from the set of all d-regular graphs on n vertices. The study of this model is often more challenging than that of $G_{n,p}$ due to the fact that the presence and absence of edges in $G_{n,d}$ are correlated. Several different techniques have been developed to deal with this model, such as the configuration model or edge-switching techniques. Robinson and Wormald [**13**] proved that $G_{n,3}$ is a.a.s. Hamiltonian, and later extended this result to $G_{n,d}$ for any fixed $d \geq 3$ [**14**]. This is in contrast to $G_{n,p}$, where the average degree must be logarithmic in n to ensure Hamiltonicity. These results were later generalised by Cooper, Frieze and Reed [**7**] and Krivelevich, Sudakov, Vu and Wormald [**9**] for the case when d is allowed to grow with n, up to $d \leq n-1$.

1.3. Local resilience

More recently, several extremal results have been translated to random graphs via the concept of local resilience. The *local resilience* of a graph G with respect to some property $\mathcal{P}$ is the maximum number $r \in \mathbb{N}$ such that, for all $H \subseteq G$ with $\Delta(H) < r$, the graph $G \smallsetminus H$ satisfies $\mathcal{P}$. We say that G is r-*resilient* with respect to a property $\mathcal{P}$ if the local resilience of G is greater than r. The systematic study of local resilience was initiated by Sudakov and Vu [**15**], and the subject has seen a lot of research since. While there are many different properties for which the study of resilience has been considered, here we concentrate on resilience with respect to Hamiltonicity.

Note that Dirac's theorem can be restated in this terminology to say that the local resilience of the complete graph K_n with respect to Hamiltonicity is $\lfloor n/2 \rfloor$. This concept of local resilience then naturally suggests a generalisation of Dirac's theorem to random graphs. In the binomial model, Lee and Sudakov [**10**] showed that, for any constant $\varepsilon > 0$, if $p \geq C \log n/n$ and C is sufficiently large, then a.a.s. $G_{n,p}$ is $(1/2 - \varepsilon)np$-resilient. This improved on earlier bounds [**2, 3, 8, 15**]. Very recently, Montgomery [**11**] and independently Nenadov, Steger and Trujić [**12**] proved a hitting time result for the local resilience of $G_{n,p}$ with respect to Hamiltonicity.

The resilience of random regular graphs with respect to Hamiltonicity is less understood. Ben-Shimon, Krivelevich and Sudakov [**2**] proved that, for large (but constant) d, a.a.s. $G_{n,d}$ is $(1-\varepsilon)d/6$-resilient with respect to Hamiltonicity. They conjectured that the true value should be closer to $d/2$.

Conjecture 1.1 (Ben-Shimon, Krivelevich and Sudakov [**2**])**.** For every $\varepsilon > 0$ there exists an integer $D = D(\varepsilon) > 0$ such that, for every fixed integer $d > D$, the local resilience of $G_{n,d}$ with respect to Hamiltonicity a.a.s. lies in the interval $((1/2-\varepsilon)d, (1/2+\varepsilon)d)$.

They also suggested to study the same problem when d is allowed to grow with n. In this direction, it is already known that for $d \gg \log n$ the random graph $G_{n,d}$ is a.a.s. $(1/2-\varepsilon)d$-resilient with respect to Hamiltonicity, and this follows by combining several results of different authors (details can be found in [**5**]).

2. Results

2.1. Binomial random graphs

In the setting of the binomial random graph, recall that Lee and Sudakov [**10**] proved an analogue of Dirac's theorem for $G_{n,p}$. Indeed, their result can be stated as follows:

For every $\varepsilon > 0$ there exists $C > 0$ such that, for $p \geq C\log n/n$, a.a.s. every subgraph G of $G_{n,p}$ with $\delta(G) \geq (1/2+\varepsilon)np$ is Hamiltonian.

Lee and Sudakov [**10**] also asked for a characterisation of the degree sequences for which the random graph $G_{n,p}$ is resilient with respect to Hamiltonicity, for p close to $\log n/n$. We partially answer this question by extending Pósa's theorem to the setting of random graphs.

Theorem 2.1 ([**6**])**.** *For every $\varepsilon > 0$, there exists $C > 0$ such that, for $p \geq C\log n/n$, a.a.s. every subgraph G of the random graph $G_{n,p}$ with degree sequence $d_1 \leq \cdots \leq d_n$ satisfying $d_i \geq (i+\varepsilon n)p$ for all $i < n/2$ is Hamiltonian.*

In a similar way, Chvátal's generalisation of Dirac's theorem suggests a similar generalisation in the setting of local resilience of random graphs. However, we prove that random graphs do not satisfy this, in a strong way.

Theorem 2.2 ([**6**])**.** *For every $0 < \varepsilon < 10^{-6}$ there exists $C > 0$ such that, for $C\log n/n \leq p \leq 1/20$, a.a.s. the following holds: not every subgraph G of $G_{n,p}$ with degree sequence $d_1 \leq \cdots \leq d_n$ satisfying that $d_i \geq (i+\varepsilon n)p$ or $d_{n-i} \geq (n-i+\varepsilon n)p$ for all $i < n/2$ contains an $\lfloor n/2 \rfloor$-matching.*

Indeed, we can construct many subgraphs of a typical instance of $G_{n,p}$ with the right degree sequence which do not contain an $\lfloor n/2 \rfloor$-matching (and hence are not Hamiltonian). The construction of these subgraphs, however, suggests that a resilience version of Chvátal's theorem may be possible if the degree sequence is 'shifted'. As evidence that this is true, we provide the following result about perfect matchings.

Theorem 2.3 ([**6**]). *For every $\varepsilon > 0$, there exists $C > 0$ such that, for $p \geq C \log n/n$, a.a.s. every subgraph G of the random graph $G_{n,p}$ with degree sequence $d_1 \leq \cdots \leq d_n$ satisfying that $d_i \geq (i + \varepsilon n)p$ or $d_{n-i-\varepsilon n} \geq (n - i + \varepsilon n)p$ for all $i < n/2$ contains a perfect matching if n is even.*

We conjecture that every subgraph of $G_{n,p}$ satisfying the conditions in Theorem 2.3 should be Hamiltonian as well.

2.2. Random regular graphs

In the setting of random regular graphs, we completely resolve Conjecture 1.1, as well as its extension to d growing slowly with n (recall that the case when $d \gg \log n$ is covered by earlier results). This can be seen as a version of Dirac's theorem for random regular graphs. Our main result gives the lower bound in Conjecture 1.1.

Theorem 2.4 ([**5**]). *For every $\varepsilon > 0$ there exists D such that, for every $D < d \leq \log^2 n$, the random graph $G_{n,d}$ is a.a.s. $(1/2 - \varepsilon)d$-resilient with respect to Hamiltonicity.*

The upper bound in Conjecture 1.1 is well-known and follows from edge distribution properties of random regular graphs. While we do not try to optimise the dependency of D on ε, we remark that D in Theorem 2.4 can be taken to be polynomial in ε^{-1}. This is essentially best possible in the sense that Theorem 2.4 fails if $d \leq (2\varepsilon)^{-1}$.

Theorem 2.5 ([**5**]). *For any odd $d > 2$, the random graph $G_{n,d}$ is not a.a.s. $(d-1)/2$-resilient with respect to Hamiltonicity.*

Our proof also shows that $G_{n,d}$ is not a.a.s. $(d-1)/2$-resilient with respect to the containment of a perfect matching. It would be interesting to obtain bounds on the resilience for small values of d.

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