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RAMSEY UPPER DENSITY OF INFINITE GRAPHS

A. LAMAISON

Abstract. Let H be an infinite graph. In a two-coloring of the edges of the complete graph on the natural numbers, what is the densest monochromatic subgraph isomorphic to H that we are guaranteed to find? We measure the density of a subgraph by the upper density of its vertex set. This question, in the particular case of the infinite path, was introduced by Erdős and Galvin. Following a recent result for the infinite path, we present bounds on the maximum density for other choices of H, including exact values for a wide class of bipartite graphs.

Let H be a graph on a countably infinite vertex set, and let $K_{\mathbb{N}}$ denote the complete graph on the natural numbers. A well-known result by Ramsey [**7**] states that every red-blue coloring of the edges of $K_{\mathbb{N}}$ contains a monochromatic infinite clique, and thus in particular there exists a monochromatic subgraph $H' \subseteq K_{\mathbb{N}}$ which is isomorphic to H. However, there are colorings of $K_{\mathbb{N}}$ in which the monochromatic infinite cliques are arbitrarily sparse. In fact, the monochromatic cliques can all have zero density, according to the following definition:

Definition 1. Let $S \subset \mathbb{N}$ be a set. We define its upper density as

$$\rho(S) = \limsup_{n\to\infty} \frac{|S \cap [n]|}{n}.$$

We want to find such a monochromatic subgraph H' in which $V(H')$ is as dense as possible. The extremal question would be what is the maximum density that we can always find. We define the parameter $\rho(H)$ accordingly:

Definition 2. Let H be a graph on a countably infinite vertex set. We define its Ramsey upper density $\rho(H)$ as the supremum of the values λ satisfying that, in every two-coloring of $K_{\mathbb{N}}$, there exists a monochromatic $H' \subseteq K_{\mathbb{N}}$ isomorphic to H in which $\rho(V(H')) \geq \lambda$.

The study of this parameter was introduced by Erdős and Galvin [**5**] for the case of the one-way infinite path P, where they showed that $2/3 \leq \rho(P) \leq 8/9$. After some improvements in [**3**] and [**6**], Corsten, DeBiasio, Lang and the author determined in [**2**] the exact value of $\rho(P)$, which is $(12 + \sqrt{8})/17 \approx 0.87226$. The case for general H was introduced by DeBiasio and McKenney in [**3**].

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Here we will generalize the techniques of [**2**] to determine or bound $\rho(H)$ for more general choices of H. We will consider that all the graphs H are countably infinite and locally finite (every vertex has finite degree).

We begin with an upper bound, which is obtained by analyzing the maximum upper density of monochromatic copies of H in a particular explicit coloring of $K_{\mathbb{N}}$. This (somewhat complicated) coloring is similar to the one described in [**2**], and depends on a real parameter $q > 1$. If we take the value of q that produces the minimum density of the copy H', we observe that the density depends on the parameter $\mu(H, n)$ that we now define.

Given a set $S \subseteq V(H)$, we define $N(S)$ to be the union of the neighborhoods of the elements of H. For any positive integer n, let $\mu(H, n)$ be the minimum value of $|N(I)|$, where I is an independent set of size n.

Theorem 1. *Let H be a graph. Then*

$$\rho(H) \leq \limsup_{n\to\infty} f\left(\frac{\mu(H,n)}{n}\right),$$

where

$$f(x) = \begin{cases} \dfrac{2x^2 + 3x + 7 + 2\sqrt{x+1}}{4x^2 + 4x + 9} & \text{for} \quad 0 \leq x < 3, \\[2ex] \dfrac{x+1}{2x} & \text{for} \quad x \geq 3. \end{cases} \tag{1}$$

For example, in the case of $P = v_1 v_2 v_3 \dots$, the independent set of size n with the smallest neighborhood is $\{v_1, v_3, \dots, v_{2n-1}\}$, whose neighborhood has size n. This produces the bound $\rho(P) \leq f(1) = (12 + \sqrt{8})/17$, the same as the bound in [**2**].

If $\limsup \frac{\mu(H,n)}{n} < 3$, the value of $f(x)$ comes from a particular choice of the parameter q. However, if $\limsup \frac{\mu(H,n)}{n} \geq 3$, then the following construction, which can be seen as the limit of the previous one when q tends to 1, is considered instead. Color the edge $ij \in K_{\mathbb{N}}$ red if $\min\{i, j\}$ is odd, and blue if it is even. If $H' \subset K_{\mathbb{N}}$ is a monochromatic (say red) copy of H, then observe that the even vertices of H' form an independent set, and that the neighborhood in H' of any set $S \subseteq V(H') \cap \{2, 4, \dots, 2t\}$ has size at most t. From this, a simple computation gives the bound of Theorem 1.

For bipartite graphs H, we can similarly find a lower bound based on the sizes of neighborhoods of independent sets. Unlike in Theorem 1, we need infinitely many equally-sized independent sets in order to use this bound.

Theorem 2. *For every bipartite graph H we have $\rho(H) \geq 1/2$. Moreover, let n be a positive integer, and λ be a positive real number. Suppose that there exist infinitely many pairwise disjoint non-empty independent sets I_i, each of size at most n, such that $|N(I_i)| \leq \lambda |I_i|$. Then $\rho(H) \geq f(\lambda)$, where f is the function in* (1).

For the case of P, for a fixed value of n, we can take infinitely many pairwise disjoint independent sets, each of the form $\{v_k, v_{k+2}, \dots, v_{k+2(n-1)}\}$, whose neighborhoods have size at most $n+1$. Therefore, taking $\lambda_n = \frac{n+1}{n}$, Theorem 2 gives $\rho(P) \geq f(\lambda_n)$. As this holds for every n, we have $\rho(P) \geq f(1)$, the same as in [**2**].

P is far from the only graph for which the supremum of the lower bounds that can be obtained from Theorem 2 equals the upper bound from Theorem 1.

Corollary 1. *The bound from* Theorem 1 *is tight in the following cases:*

- *if H is a forest.*
- *if H is a bipartite graph in which every orbit of the automorphism group acting on the vertex set is infinite.*

In particular, given a finite graph F, denote by $\omega \cdot F$ the disjoint union of countably infinitely many copies of F. If F is bipartite, then

$$\rho(\omega \cdot F) = f\left(\min_{\substack{\emptyset \neq I \subseteq V(F) \\ I \text{ indep.}}} \frac{|N(I)|}{|I|} \right).$$

For example, if $F = C_{2r}$, the independent sets that minimize $\frac{|N(I)|}{|I|}$ are the two color classes in the bipartition of F, which satisfy $|N(I)| = r = |I|$, hence $f(\omega \cdot C_{2r}) = f(1)$.

In the case of non-bipartite graphs, there are more factors that play a role in the value of $\rho(H)$, such as the chromatic number and the number of components of H in which the 'good' independent sets are located. For example, consider the graph $P + K_3$, the disjoint union of an infinite path and a triangle. Consider the coloring of $K_{\mathbb{N}}$ in which ij is red if and only if $i+j$ is odd. The red edges form a bipartite graph, so any monochromatic copy of $P+K_3$ must be blue. But the blue graph consists of two components, each with upper density 1/2, and the infinite path must be contained in one of them (the triangle does not contribute to the upper density). This means that $\rho(P+K_3) \leq 1/2$, despite having essentially the same independent sets as P. For two infinite paths and a triangle, however, we will show that $\rho(2 \cdot P + K_3) = f(1)$.

We say that a set $I \subseteq V(H)$ is doubly independent if both I and $N(I)$ are independent. The following theorem gives a lower bound for some graphs in terms of their doubly independent sets:

Theorem 3. *Let H be a graph, n, k be positive integers and λ be a positive real number. Let $\chi\colon V(H) \to [k]$ be a proper coloring. Suppose that, whenever we remove at most $k-2$ components from H, there are still infinitely many pairwise disjoint non-empty doubly independent sets I_i, each of size at most n, such that $|N(I_i)| \leq \lambda |I_i|$ and $N(I_i)$ is monochromatic in χ. Then $\rho(H) \geq f(\lambda)$, where f is the function in* (1).

For the case $H = 2 \cdot P + K_3$, we can color the triangle in colors $\{1,2,3\}$ and the paths with colors $\{1,2\}$ to obtain a proper coloring of H. Then in both paths the neighborhood of each independent set $\{v_k, v_{k+2}, \dots, v_{k+2(n-1)}\}$ is monochromatic. As in the case of P, this leads to $\rho(2 \cdot P + K_3) \geq f(1)$, which matches the upper bound $\rho(2 \cdot P + K_3) \leq f(1)$ that we obtain from Theorem 1

While there are non-bipartite F for which Theorems 1 and 3 give the same bound for $\rho(\omega \cdot F)$, namely those in which an independent set that minimizes $\frac{|N(I)|}{|I|}$ is doubly independent (see figure), this is not usually the case. However, this result is useful for the asymptotic behavior of some families of graphs. For example, by Corollary 1 we have $\rho(\omega \cdot C_{2r}) = f(1)$ for every $r \geq 2$. For odd cycles, the Ramsey density approaches $f(1)$:

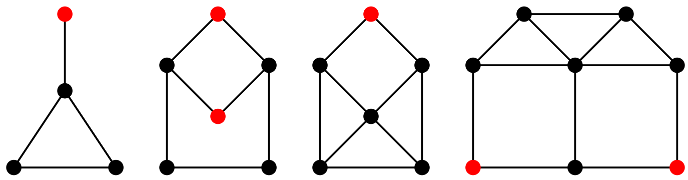

Figure 1. Four non-bipartite graphs F for which Theorems 1 and 3 give the same bound on $\rho(\omega \cdot F)$, with their doubly independent sets indicated.

Corollary 2.

$$\lim_{r \to \infty} \rho(\omega \cdot C_{2r+1}) = f(1)$$

The coloring used to prove the lower bound of Corollary 2 uses three colors. Every cycle $v_1 v_2 \ldots v_{2r+1}$ has v_{2r+1} in color 3 and the rest of the cycle alternates colors 1 and 2. The set $I = \{v_2, v_4, \ldots, v_{2r-2}\}$ is doubly independent, its neighborhood is monochromatic and $\frac{|N(I)|}{|I|} = \frac{r}{r-1}$.

If every vertex of H is contained in a triangle, then the neighborhood of every vertex contains an edge, so no non-empy doubly independent set exists, and thus no lower bound on $\rho(H)$ can be deduced from Theorem 3. For example, this happens in $\omega \cdot K_3$, or in powers of P. In order to study these graphs we need approaches different to those from the previous results.

For graphs of the form $\omega \cdot F$, we can use the Ramsey number of $k \cdot F$. Let $r(n \cdot F)$ denote the two-color Ramsey number of $k \cdot F$, that is, the minimum n such that in every two-coloring of K_n there are k disjoint monochromatic copies of F, all in the same color. The asymptotic behavior of $r(k \cdot n)$ was determined by Burr, Erdős and Spencer [**1**].

Theorem 4. *Let F be a finite graph. Then*

$$\rho(\omega \cdot F) \geq \lim_{k \to \infty} \frac{k|V(F)|}{r(k \cdot F)} = \frac{|V(F)|}{2|V(F)| - \alpha(F)}.$$

For $\omega \cdot K_3$, Theorem 4 gives best known lower bound, which is $\rho(\omega \cdot K_3) \geq 3/5$. In contrast the best known upper bound comes from Theorem 1, and gives $\rho(\omega \cdot K_3) \leq f(2) \approx 0.74133$.

Another approach is to use a decomposition of Elekes, D. Soukup, L. Soukup and Szentmiklóssy [**4**]. Given a coloring of $K_{\mathbb{N}}$, we say that a set $S \subseteq \mathbb{N}$ is k-red-staged if there exist sets $S_1, S_2, \ldots, S_k \subseteq \mathbb{N}$, not necessarily different, such that

$S_1 = S$ and, for every $1 \leq t \leq k-1$ and every finite subset $T \subseteq \cup_{i=1}^{t} S_i$, the intersection of the red neighborhoods of the vertices of T in S_{t+1} is infinite.

The k-th power of a graph G is a graph G^k on the same vertex set, where two vertices are adjacent in G^k if and only if they are at distance at most k in G. [**4**] proves that $K_{\mathbb{N}}$ can be partitioned into at most 2^{2k-1} sets, each of which is k-red-staged or k-blue-staged, and then uses the partition to show that in any red-blue coloring of $K_{\mathbb{N}}$ the vertex set can be partitioned into at most 2^{2k-1} monochromatic copies of P^k and a finite set. As noted in [**3**], this implies $\rho(P^k) \geq 2^{1-2k}$.

DeBiasio and McKenney conjectured in [**3**] that for every positive integer k there exists $\epsilon > 0$ such that $\rho(H) > \epsilon$ for every graph H with maximum degree at most k. Our next result proves this conjecture, and improves the bound on $\rho(P^k)$:

Theorem 5. *Let H be a graph with chromatic number k in which, whenever we remove at most $t-1$ components, infinitely many vertices remain. Then*

$$\rho(H) \geq \min\left\{\frac{t}{2k-2}, \frac{1}{2}\right\}.$$

Furthermore, if t components can be removed in such a way that only finitely many vertices remain then

$$\rho(H) \leq \min\left\{\frac{t}{k-1}, 1\right\}.$$

Any graph H with maximum degree k has chromatic number at most $k+1$, so by Theorem 5 with $t = 1$ we have $\rho(H) \geq \frac{1}{2k}$. The chromatic number of P^k is $k+1$, so again setting $t = 1$ we obtain $\rho(P^k) \geq \frac{1}{2k}$, which is much larger than 2^{1-2k}.

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A. Lamaison, Institut für Mathematik, Freie Universität Berlin, Berlin, Germany,
e-mail: `lamaison@math.fu-berlin.de`