Abstract. A path decomposition of a graph $G$ is a collection of edge-disjoint paths of $G$ that covers the edge set of $G$. Gallai (1968) conjectured that every connected graph on $n$ vertices admits a path decomposition of cardinality at most $\lceil (n+1)/2 \rceil$. Seminal results toward its verification consider the graph obtained from $G$ by removing its vertices with odd degree, which is called the $E$-subgraph of $G$. Lovász (1968) verified Gallai’s Conjecture for graphs whose $E$-subgraphs consist of at most one vertex, and Pyber (1996) verified it for graphs whose $E$-subgraphs are forests. In 2005, Fan verified Gallai’s Conjecture for graphs whose $E$-subgraphs are triangle-free and contain only blocks with maximum degree at most 3. Since then, no result was obtained regarding $E$-subgraphs. In this paper, we verify Gallai’s Conjecture for graphs whose $E$-subgraphs have maximum degree at most 3.

1. Introduction

In this paper, all graphs considered are finite and simple, i.e., contain a finite number of vertices and edges and have neither loops nor multiple edges. The terminology and notation used in this paper are standard (see, e.g. [3]). We say that a vertex is even (resp. odd) if it has even (resp. odd) degree. A path decomposition $\mathcal{D}$ of a graph $G$ is a collection of edge-disjoint paths of $G$ that covers all the edges of $G$. A path decomposition $\mathcal{D}$ of a graph $G$ is minimum if for every path decomposition $\mathcal{D}'$ of $G$ we have $|\mathcal{D}| \leq |\mathcal{D}'|$, and the cardinality of such a minimum path decomposition, denoted by $\text{pn}(G)$, is called the path number of $G$. In 1968, Gallai proposed the following conjecture (see [2, 11]).

**Conjecture 1** (Gallai, 1968). If $G$ is a connected graph, then $\text{pn}(G) \leq \left\lceil \frac{|V(G)|}{2} \right\rceil$.

Lovász [11] verified Conjecture 1 for graphs that have at most one even vertex. Pyber [12] extended Lovász’s result by proving that Conjecture 1 holds for graphs in which each cycle contains at least one odd vertex. In 2005, Fan [7] extended these results by extending Lovász’s technique, and exploring the following special structure. Given a graph $G$, the $E$-subgraph of $G$, denoted by $EV(G)$, is the graph...
obtained from $G$ by removing its odd vertices, or, equivalently, the subgraph of $G$ induced by its even vertices. Thus, the results above may be restated as follows.

**Theorem 2** (Lovász, 1968; Pyber, 1996; Fan, 2005). Let $G$ be a connected graph on $n$ vertices. Then, the following hold.

(a) If $EV(G)$ contains at most one vertex, then $pn(G) \leq \lfloor n/2 \rfloor$;
(b) If $EV(G)$ is a forest, then $pn(G) \leq \lfloor n/2 \rfloor$; and
(c) If each block of $EV(G)$ is triangle-free and has maximum degree at most 3, then $pn(G) \leq \lfloor n/2 \rfloor$.

Given an even vertex $v$ in $G$, the $E$-degree of $v$ is the degree of $v$ in $EV(G)$.

Let $G_3$ denote the set of connected graphs in which its even vertices have $E$-degree at most 3, i.e., $G_3 = \{G \text{ is a connected graph such that } \Delta(EV(G)) \leq 3\}$. In this paper we give a step further toward strengthening the result in [7], by presenting a strategy to deal with triangles of the $E$-subgraph, and verifying Conjecture 1 for graphs in $G_3$. Due to space limitations, we present only a sketch of the proof.

**Theorem 3.** If $G \in G_3$, then $pn(G) \leq \lceil |V(G)|/2 \rceil$.

Conjecture 1 has been deeply explored, and the literature indicating its correctness include results for Eulerian graphs with maximum degree at most 4 [8]; a family of regular graphs [5]; a family of triangle-free graphs [10]; and maximal outerplanar graphs and 2-connected outerplanar graphs [9]. Recent results were obtained by Bonamy and Perrett [1] who verified Conjecture 1 for graphs with maximum degree at most 5.

Note that the results in Theorem 2 give a bound of $\lfloor n/2 \rfloor$ for the graphs studied, which is slightly different from the bound of $\lceil n/2 \rceil$ proposed by Gallai. A trivial condition for a graph on $n$ vertices not to admit the former bound is to have sufficiently many edges. More precisely, if $|E(G)| > \lfloor n/2 \rfloor(n-1)$, then we have $pn(G) \geq \lfloor n/2 \rfloor$. In this case, $n$ must be an odd integer. Such graphs are known as odd semi-cliques [1]. This motivates the following strengthening of Conjecture 1.

**Conjecture 4.** If $G$ is a connected graph, then either $pn(G) \leq \lfloor |V(G)|/2 \rfloor$ or $G$ is an odd semi-clique.

Botler, Coelho, Lee, and Sambinelli [4] verified Conjecture 4 for graphs with treewidth at most 3 by proving that a partial 3-tree with $n$ vertices either has path number at most $\lfloor n/2 \rfloor$, which are called Gallai graphs, or is one of the two odd semi-cliques that are partial 3-trees ($K_3$ and $K_3 - e$). They also prove [4] an analogous result for graphs with maximum degree at most 4. More recently, Botler, Jiménez, and Sambinelli [6] verified Conjecture 4 for triangle-free planar graphs by proving that every such graph is a Gallai graph.

Finally, we note that the results obtained so far deal with classes of graphs that contain only a finite number of odd semi-cliques. This is not the case of $G_3$, since $K_{2k+1} \setminus M_{k-1}$, the graph obtained from a complete graph with $2k+1$ vertices by removing a matching $M_{k-1}$ of size $k-1$, is an odd semi-clique and belongs to $G_3$, for any $k \in \mathbb{N}$.
2. Technical Lemmas

In this section we present some technical results used in our proof. We use two lemmas presented by Fan [7, Lemma 3.4 and Lemma 3.6]. Given a path decomposition \( D \), we denote by \( D(v) \) the number of paths in \( D \) having the vertex \( v \) as an end vertex. Following the strategy presented by Fan, our technique is based in the following definition.

**Definition 1.** Let \( D \) be a graph and let \( B \) be a set of edges incident to \( a \). Let \( G' = G \setminus B \), and let \( D' \) be a path decomposition of \( G' \). We say that a subset \( A = \{ax_i: 1 \leq i \leq k\} \) of \( B \) is addible at \( a \) with respect to \( D' \) if \( G' + A \) has a path decomposition \( D' \) such that \( |D'| = |D'| \) and

(a) \( D(a) = D'(a) + |A| \) and \( D(x_i) = D'(x_i) - 1 \), for \( 1 \leq i \leq k \); and

(b) \( D(v) = D'(v) \) for each \( v \in V(G) \setminus \{a, x_1, \ldots, x_k\} \).

Moreover, we say that \( D \) is a transformation of \( D' \) by adding \( A \) at \( a \). When \( k = 1 \), we simply say that \( ax_1 \) is addible at \( a \) with respect to \( D' \).

The next lemma presents conditions for an edge (or a set of edges) to be addible.

**Lemma 5** (Fan, 2005).

(a) Let \( G \) be a graph and \( ab \in E(G) \). Suppose that \( D' \) is a path decomposition of \( G' = G \setminus ab \). If \( D'(b) > |\{v \in N_{G'}(a): D'(v) = 0\}| \), then \( ab \) is addible at \( a \) with respect to \( D' \).

(b) Let \( a \) be a vertex in a graph \( G \) and let \( G' = G \setminus \{ax_1, \ldots, ax_h\} \), where \( x_i \in N_{G'}(a) \). Suppose that \( D' \) is a path decomposition of \( G' \) with \( D'(v) \geq 1 \) for every \( v \in N_{G'}(a) \). Then, for any \( x \in \{x_1, \ldots, x_h\} \), there is \( B \subseteq \{ax_1, \ldots, ax_h\} \) such that \( ax \in B \), \( |B| \geq \left\lceil \frac{h}{2} \right\rceil \), and \( B \) is addible at \( a \) with respect to \( D' \).

3. Main theorem

We say that a graph \( G \) is a single even triangle graph (SET graph) if \( EV(G) \cong K_3 \), and every odd vertex of \( G \) has at least two even neighbors. Note that every SET graph has an odd number of vertices. We can obtain a decomposition \( D' \) of a SET graph \( G \) such that \( |D'| \leq (|V(G)| + 1)/2 \) as follows. Let \( e \in E(EV(G)) \) and \( G' = G \setminus e \), and let \( D' \) be a minimum path decomposition of \( G' \). By Theorem 2(b), \( |D'| \leq (|V(G)| - 1)/2 \), and hence \( D' \cup \{e\} \) is the desired decomposition.

Our main theorem is a weaker version of Conjecture 4 in which we replace odd semi-cliques by SET graphs. The proof of our main theorem consists in showing that, for a minimal counterexample \( G \), the graph \( EV(G) \) consists of disjoint triangles in which no odd vertex is adjacent to two of these triangles. Then, we remove the edges of a special subgraph joining two of these triangles, yielding a suitable proper subgraph of \( G \) from which we obtain a good decomposition of \( G \).

**Theorem 6.** If \( G \in \mathcal{G}_3 \), then \( G \) is either a Gallai graph or a SET graph.

**Sketch of the proof.** Suppose that the statement does not hold, and let \( G \in \mathcal{G}_3 \) be a counterexample minimizing \( |E(G)| \). Let \( n = |V(G)| \). In what follows, we state
a few claims regarding $G$. The following claim is obtained by applying Lemma 5 on even vertices.

**Claim 1.** No vertex of $G$ has exactly one even neighbor, and every component of $EV(G)$ is a triangle or an isolated vertex.

In what follows, if $x$ is an odd vertex and $T \subseteq EV(G)$ is a triangle containing a neighbor of $x$ in $G$, then we say that $T$ is a triangle neighbor of $x$. The proof of the next claim consists in extending Fan's techniques for odd vertices.

**Claim 2.** If $v$ is an odd vertex in $G$, then $v$ has neighbors in at most one component of $EV(G)$.

Note that Claim 2 implies that two even vertices have a common odd neighbor only if they belong to the same (triangle) component of $EV(G)$. Now, suppose that $x$ is an isolated vertex in $EV(G)$, and let $y$ be a neighbor of $x$ in $G$. By Claim 2, $x$ is the only even neighbor of $y$, a contradiction to Claim 1.

Given a vertex $v$ of $G$ that has a triangle neighbor $T$, we say that $v$ is a full vertex if every vertex of $T$ is a neighbor of $v$.

**Claim 3.** Let $v$ be a vertex of $G$ that has a triangle neighbor $T$. Then,

(a) If $v$ has an odd neighbor that has no even neighbor, then $v$ is a full vertex;
(b) Every odd neighbor of $v$ has an even neighbor; and
(c) If $u$ is an odd neighbor of $v$ with a triangle neighbor different from $T$, then $u$ and $v$ are full vertices.

First, suppose that $EV(G) \cong K_3$. By Claim 3(b), every odd vertex of $G$ has an even neighbor, and hence, by Claim 1, every odd vertex of $G$ has at least two even neighbors. Thus, $G$ is a SET graph. Therefore, we may assume that $EV(G)$ has at least two components. Thus, let $P$ be a shortest path joining vertices of two different components of $EV(G)$. It follows from Claims 1, 2, and 3(b) that $P$ contains precisely two internal vertices, say $u$ and $v$. Let $T_u$ (resp. $T_v$) be the triangle neighbor of $u$ (resp. $v$).

Let $V(T_u) = \{a, b, c\}$ and $V(T_v) = \{x, y, z\}$. Let $S_u = \{uw \in E(G) : w \in V(T_u)\}$, $S_v = \{vw \in E(G) : w \in V(T_v)\}$, and $G_0 = G \setminus \{(uv) \cup S_u \cup S_v\}$. By Claim 3(c), the vertices $u$ and $v$ are full vertices, i.e., $|S_u| = |S_v| = 3$, and hence $w$ is odd in $G_0$ for every $w \in V(T_u) \cup V(T_v) \cup \{u, v\}$. Therefore, $G_0 \in \mathcal{G}_3$. Moreover, we can prove that no component of $G_0$ is a SET graph, and hence $G_0$ is a Gallai graph. Let $\mathcal{D}_0$ be a minimum path decomposition of $G_0$. By the minimality of $G$, we have $|\mathcal{D}_0| \leq \lfloor n/2 \rfloor$. In what follows, we obtain a path decomposition $\mathcal{D}_3$ of $G_3 = G_0 + uv + S_u = G \setminus S_v$ such that $\mathcal{D}_3(u), \mathcal{D}_3(v) \geq 1$.

First, we obtain a path decomposition $\mathcal{D}_2$ of $G_2 = G_0 + S_u$ such that $\mathcal{D}_2(u) \geq 2$. By Lemma 5(b), there is a $B_u \subseteq S_u$ such that $|B_u| \geq \lfloor |S_u|/2 \rfloor$ and $B_u$ is addible at $u$ with respect to $\mathcal{D}_0$. Let $\mathcal{D}_1$ be the transformation of $\mathcal{D}_0$ by adding $B_u$ at $u$. We have $\mathcal{D}_1(v) \geq 1 + \lceil |S_v|/2 \rceil$. Note that $S_u \setminus B_u$ contains at most one edge. If $S_u \setminus B_u = \emptyset$, then put $\mathcal{D}_2 = \mathcal{D}_1$ is the desired decomposition. If $S_u \setminus B_u \neq \emptyset$, then suppose $wc \in S_u \setminus B_u$ and put $G_1 = G_0 + B_u$. Note that $\{x \in N_G(c) : \mathcal{D}_1(x) = 0\} \subseteq \{a, b\}$. By Lemma 5(a), $wc$ is addible at $c$ with respect to $\mathcal{D}_1$. Then, the
transformation $D_2$ of $D_1$ by adding $uc$ at $c$ is the desired decomposition. Now, note that every neighbor of $v$ in $G_2 = G_1 + uc = G_0 + S_u$ is odd, and hence, by Lemma 5(a), $uv$ is addible at $v$ with respect to $D_2$. Then, the transformation $D_3$ of $D_2$ by adding $uv$ at $v$ is a path decomposition of $G_3 = G_0 + uv + S_u = G \setminus S_v$ such that $D_3(u), D_3(v) \geq 1$. Analogously, we obtain a transformation $D_4$ of $D_3$ by adding $S_v$. Since $|D_4| \leq \lfloor n/2 \rfloor$, $G$ is a Gallai graph. This concludes the proof. □

4. FUTURE WORKS

The result in this paper may be extended in two natural directions: (1) extending Theorem 6 to a strengthening of Theorem 2(c) in which we remove the triangle-free condition, i.e., by verifying (a strengthening of) Conjecture 1 for graphs in which each block of its E-subgraph has maximum degree at most 3; and (2) Replacing SET graphs by odd semi-cliques, which verifies Conjecture 4 for graphs in $\mathcal{G}_3$. Further, the techniques used in this paper may be combined with reducing schemes (see [6]) in order to extend previous results.

REFERENCES


F. Botler, Programa de Engenharia de Sistemas e Computação, Universidade Federal do Rio de Janeiro, Brazil, e-mail: fbotler@cos.ufrj.br

M. Sambinelli, Instituto de Matemática e Estatística, Universidade de Sao Paulo, Brazil, e-mail: sambinelli@ime.usp.br