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GALLAI'S PATH DECOMPOSITION CONJECTURE FOR GRAPHS WITH MAXIMUM E-DEGREE AT MOST 3

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ABSTRACT. A path decomposition of a graph G is a collection of edge-disjoint paths of G that covers the edge set of G. Gallai (1968) conjectured that every connected graph on n vertices admits a path decomposition of cardinality at most $\lfloor (n+1)/2 \rfloor$. Seminal results toward its verification consider the graph obtained from G by removing its vertices with odd degree, which is called the *E-subgraph* of G. Lovász (1968) verified Gallai's Conjecture for graphs whose E-subgraphs consist of at most one vertex, and Pyber (1996) verified it for graphs whose E-subgraphs are forests. In 2005, Fan verified Gallai's Conjecture for graphs whose E-subgraphs are triangle-free and contain only blocks with maximum degree at most 3. Since then, no result was obtained regarding E-subgraphs. In this paper, we verify Gallai's Conjecture for graphs whose E-subgraphs have maximum degree at most 3.

1. INTRODUCTION

In this paper, all graphs considered are finite and simple, i.e., contain a finite number of vertices and edges and have neither loops nor multiple edges. The terminology and notation used in this paper are standard (see, e.g. [**3**]). We say that a vertex is *even* (resp. *odd*) if it has even (resp. *odd*) degree. A *path decomposition* $\mathcal{D}$ of a graph G is a collection of edge-disjoint paths of G that covers all the edges of G. A path decomposition $\mathcal{D}$ of a graph G is *minimum* if for every path decomposition $\mathcal{D}'$ of G we have $|\mathcal{D}| \leq |\mathcal{D}'|$, and the cardinality of such a minimum path decomposition, denoted by $\mathrm{pn}(G)$, is called the *path number* of G. In 1968, Gallai proposed the following conjecture (see [**2, 11**]).

Conjecture 1 (Gallai, 1968)**.** If G is a connected graph, then $\mathrm{pn}(G) \leq \left\lceil \frac{|V(G)|}{2} \right\rceil$.

Lovász [**11**] verified Conjecture 1 for graphs that have at most one even vertex. Pyber [**12**] extended Lovász's result by proving that Conjecture 1 holds for graphs in which each cycle contains at least one odd vertex. In 2005, Fan [**7**] extended these results by extending Lovász's technique, and exploring the following special structure. Given a graph G, the *E-subgraph* of G, denoted by $EV(G)$, is the graph

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obtained from G by removing its odd vertices, or, equivalently, the subgraph of G induced by its even vertices. Thus, the results above may be restated as follows.

Theorem 2 (Lovász, 1968; Pyber, 1996; Fan, 2005)**.** *Let G be a connected graph on n vertices. Then, the following hold.*

(a) *If $EV(G)$ contains at most one vertex, then* $\mathrm{pn}(G) \leq \lfloor n/2 \rfloor$*;*
(b) *If $EV(G)$ is a forest, then* $\mathrm{pn}(G) \leq \lfloor n/2 \rfloor$*; and*
(c) *If each block of $EV(G)$ is triangle-free and has maximum degree at most* 3*, then* $\mathrm{pn}(G) \leq \lfloor n/2 \rfloor$*.*

Given an even vertex v in G, the *E-degree* of v is the degree of v in $EV(G)$. Let $\mathcal{G}_3$ denote the set of connected graphs in which its even vertices have E-degree at most 3, i.e., $\mathcal{G}_3 = \big\{G \text{ is a connected graph such that } \Delta\big(EV(G)\big) \leq 3\big\}$. In this paper we give a step further toward strengthening the result in [**7**], by presenting a strategy to deal with triangles of the E-subgraph, and verifying Conjecture 1 for graphs in $\mathcal{G}_3$. Due to space limitations, we present only a sketch of the proof.

Theorem 3. *If $G \in \mathcal{G}_3$, then* $\mathrm{pn}(G) \leq \big\lceil |V(G)|/2 \big\rceil$*.*

Conjecture 1 has been deeply explored, and the literature indicating its correctness include results for Eulerian graphs with maximum degree at most 4 [**8**]; a family of regular graphs [**5**]; a family of triangle-free graphs [**10**]; and maximal outerplanar graphs and 2-connected outerplanar graphs [**9**]. Recent results were obtained by Bonamy and Perrett [**1**] who verified Conjecture 1 for graphs with maximum degree at most 5.

Note that the results in Theorem 2 give a bound of $\lfloor n/2 \rfloor$ for the graphs studied, which is slightly different from the bound of $\lceil n/2 \rceil$ proposed by Gallai. A trivial condition for a graph on n vertices not to admit the former bound is to have sufficiently many edges. More precisely, if $|E(G)| > \lfloor n/2 \rfloor (n-1)$, then we have $\mathrm{pn}(G) \geq \lceil n/2 \rceil$. In this case, n must be an odd integer. Such graphs are known as *odd semi-cliques* [**1**]. This motivates the following strengthening of Conjecture 1.

Conjecture 4. If G is a connected graph, then either $\mathrm{pn}(G) \leq \big\lfloor |V(G)|/2 \big\rfloor$ or G is an odd semi-clique.

Botler, Coelho, Lee, and Sambinelli [**4**] verified Conjecture 4 for graphs with treewidth at most 3 by proving that a partial 3-tree with n vertices either has path number at most $\lfloor n/2 \rfloor$, which are called *Gallai graphs*, or is one of the two odd semi-cliques that are partial 3-trees (K_3 and $K_5 - e$). They also prove [**4**] an analogous result for graphs with maximum degree at most 4. More recently, Botler, Jiménez, and Sambinelli [**6**] verified Conjecture 4 for triangle-free planar graphs by proving that every such graph is a Gallai graph.

Finally, we note that the results obtained so far deal with classes of graphs that contain only a finite number of odd semi-cliques. This is not the case of $\mathcal{G}_3$, since $K_{2k+1} \setminus M_{k-1}$, the graph obtained from a complete graph with $2k+1$ vertices by removing a matching M_{k-1} of size $k-1$, is an odd semi-clique and belongs to $\mathcal{G}_3$, for any $k \in \mathbb{N}$.

2. Technical Lemmas

In this section we present some technical results used in our proof. We use two lemmas presented by Fan [**7**, Lemma 3.4 and Lemma 3.6]. Given a path decomposition $\mathcal{D}$, we denote by $\mathcal{D}(v)$ the number of paths in $\mathcal{D}$ having the vertex v as an end vertex. Following the strategy presented by Fan, our technique is based in the following definition.

Definition 1. Let a be a vertex in a graph G and let B be a set of edges incident to a. Let $G' = G \setminus B$, and let $\mathcal{D}'$ be a path decomposition of G'. We say that a subset $A = \{ax_i : 1 \le i \le k\}$ of B is *addible at a with respect to* $\mathcal{D}'$ if $G' + A$ has a path decomposition $\mathcal{D}$ such that $|\mathcal{D}| = |\mathcal{D}'|$ and

(a) $\mathcal{D}(a) = \mathcal{D}'(a) + |A|$ and $D(x_i) = D'(x_i) - 1$, for $1 \le i \le k$; and
(b) $D(v) = D'(v)$ for each $v \in V(G) \setminus \{a, x_1, \ldots, x_k\}$.

Moreover, we say that $\mathcal{D}$ is a *transformation of* $\mathcal{D}'$ *by adding A at a.* When $k = 1$, we simply say that ax_1 is *addible at a with respect to* $\mathcal{D}'$.

The next lemma present conditions for an edge (or a set of edges) to be addible.

Lemma 5 (Fan, 2005)**.**

(a) *Let G be a graph and* $ab \in E(G)$. *Suppose that* $\mathcal{D}'$ *is a path decomposition of* $G' = G \setminus ab$. *If* $\mathcal{D}'(b) > |\{v \in N_{G'}(a) : \mathcal{D}'(v) = 0\}|$, *then ab is addible at a with respect to* $\mathcal{D}'$.
(b) *Let a be a vertex in a graph G and let* $G' = G \setminus \{ax_1, \ldots, ax_h\}$, *where* $x_i \in N_G(a)$. *Suppose that* $\mathcal{D}'$ *is a path decomposition of* G' *with* $\mathcal{D}'(v) \ge 1$ *for every* $v \in N_G(a)$. *Then, for any* $x \in \{x_1, \ldots, x_h\}$, *there is* $B \subseteq \{ax_1, \ldots, ax_h\}$ *such that* $ax \in B$, $|B| \ge \lceil \frac{h}{2} \rceil$, *and B is addible at a with respect to* $\mathcal{D}'$.

3. Main theorem

We say that a graph G is a *single even triangle graph (SET graph)* if $EV(G) \cong K_3$, and every odd vertex of G has at least two even neighbors. Note that every SET graph has an odd number of vertices. We can obtain a decomposition $\mathcal{D}$ of a SET graph G such that $|\mathcal{D}| \le (|V(G)| + 1)/2$ as follows. Let $e \in E(EV(G))$ and $G' = G \setminus e$, and let $\mathcal{D}'$ be a minimum path decomposition of G'. By Theorem 2(b), $|\mathcal{D}'| \le (|V(G)| - 1)/2$, and hence $\mathcal{D}' \cup \{e\}$ is the desired decomposition.

Our main theorem is a weaker version of Conjecture 4 in which we replace odd semi-cliques by SET graphs. The proof of our main theorem consists in showing that, for a minimal counterexample G, the graph $EV(G)$ consists of disjoint triangles in which no odd vertex is adjacent to two of these triangles. Then, we remove the edges of a special subgraph joining two of these triangles, yielding a suitable proper subgraph of G from which we obtain a good decomposition of G.

Theorem 6. *If* $G \in \mathcal{G}_3$, *then G is either a Gallai graph or a SET graph.*

Sketch of the proof. Suppose that the statement does not hold, and let $G \in \mathcal{G}_3$ be a counterexample minimizing $|E(G)|$. Let $n = |V(G)|$. In what follows, we state a few claims regarding G. The following claim is obtained by applying Lemma 5 on even vertices.

Claim 1. *No vertex of G has exactly one even neighbor, and every component of $EV(G)$ is a triangle or an isolated vertex.*

In what follows, if x is an odd vertex and $T \subseteq EV(G)$ is a triangle containing a neighbor of x in G, then we say that T is a *triangle neighbor* of x. The proof of the next claim consists in extending Fan's techniques for odd vertices.

Claim 2. *If v is an odd vertex in G, then v has neighbors in at most one component of $EV(G)$.*

Note that Claim 2 implies that two even vertices have a common odd neighbor only if they belong to the same (triangle) component of $EV(G)$. Now, suppose that x is an isolated vertex in $EV(G)$, and let y be a neighbor of x in G. By Claim 2, x is the only even neighbor of y, a contradiction to Claim 1.

Given a vertex v of G that has a triangle neighbor T, we say that v is a *full vertex* if every vertex of T is a neighbor of v.

Claim 3. *Let v be a vertex of G that has a triangle neighbor T. Then,*

(a) *If v has an odd neighbor that has no even neighbor, then v is a full vertex;*
(b) *Every odd neighbor of v has an even neighbor; and*
(c) *If u is an odd neighbor of v with a triangle neighbor different from T, then u and v are full vertices.*

First, suppose that $EV(G) \cong K_3$. By Claim 3(b), every odd vertex of G has an even neighbor, and hence, by Claim 1, every odd vertex of G has at least two even neighbors. Thus, G is a SET graph. Therefore, we may assume that $EV(G)$ has at least two components. Thus, let P be a shortest path joining vertices of two different components of $EV(G)$. It follows from Claims 1, 2, and 3(b) that P contains precisely two internal vertices, say u and v. Let T_u (resp. T_v) be the triangle neighbor of u (resp. v).

Let $V(T_u) = \{a, b, c\}$ and $V(T_v) = \{x, y, z\}$. Let $S_u = \big\{uw \in E(G) \colon w \in V(T_u)\big\}$, $S_v = \big\{vw \in E(G) \colon w \in V(T_v)\big\}$, and $G_0 = G \setminus \big(\{uv\} \cup S_u \cup S_v\big)$. By Claim 3(c), the vertices u and v are full vertices, i.e., $|S_u| = |S_v| = 3$, and hence w is odd in G_0 for every $w \in V(T_u) \cup V(T_v) \cup \{u, v\}$. Therefore, $G_0 \in \mathcal{G}_3$. Moreover, we can prove that no component of G_0 is a SET graph, and hence G_0 is a Gallai graph. Let $\mathcal{D}_0$ be a minimum path decomposition of G_0. By the minimality of G, we have $|\mathcal{D}_0| \leq \lfloor n/2 \rfloor$. In what follows, we obtain a path decomposition $\mathcal{D}_3$ of $G_3 = G_0 + uv + S_u = G \setminus S_v$ such that $\mathcal{D}_3(u), \mathcal{D}_3(v) \geq 1$.

First, we obtain a path decomposition $\mathcal{D}_2$ of $G_2 = G_0 + S_u$ such that $\mathcal{D}_2(u) \geq 2$. By Lemma 5 (b), there is a $B_u \subseteq S_u$ such that $|B_u| \geq \lceil |S_u|/2 \rceil$ and B_u is addible at u with respect to $\mathcal{D}_0$. Let $\mathcal{D}_1$ be the transformation of $\mathcal{D}_0$ by adding B_u at u. We have $\mathcal{D}_1(v) \geq 1 + \big\lceil |S|/2 \big\rceil$. Note that $S_u \setminus B_u$ contains at most one edge. If $S_u \setminus B_u = \emptyset$, then put $\mathcal{D}_2 = \mathcal{D}_1$ is the desired decomposition. If $S_u \setminus B_u \neq \emptyset$, then suppose $uc \in S_u \setminus B_u$ and put $G_1 = G_0 + B_u$. Note that $\{x \in N_{G_1}(c) \colon \mathcal{D}_1(x) = 0\} \subseteq \{a, b\}$. By Lemma 5(a), uc is addible at c with respect to $\mathcal{D}_1$. Then, the transformation $\mathcal{D}_2$ of $\mathcal{D}_1$ by adding uc at c is the desired decomposition. Now, note that every neighbor of v in $G_2 = G_1 + uc = G_0 + S_u$ is odd, and hence, by Lemma 5(a), uv is addible at v with respect to $\mathcal{D}_2$. Then, the transformation $\mathcal{D}_3$ of $\mathcal{D}_2$ by adding uv at v is a path decomposition of $G_3 = G_0 + uv + S_u = G \setminus S_v$ such that $\mathcal{D}_3(u), \mathcal{D}_3(v) \geq 1$. Analogously, we obtain a transformation $\mathcal{D}_4$ of $\mathcal{D}_3$ by adding S_v. Since $|\mathcal{D}_4| \leq \lfloor n/2 \rfloor$, G is a Gallai graph. This concludes the proof. □

4. Future works

The result in this paper may be extended in two natural directions: (1) extending Theorem 6 to a strengthening of Theorem 2(c) in which we remove the triangle-free condition, i.e., by verifying (a strengthening of) Conjecture 1 for graphs in which each block of its E-subgraph has maximum degree at most 3; and (2) Replacing SET graphs by odd semi-cliques, which verifies Conjecture 4 for graphs in $\mathcal{G}_3$. Further, the techniques used in this paper may be combined with reducing schemes (see [**6**]) in order to extend previous results.

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