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# ON SOME SUBMANIFOLDS OF GENERALIZED $(\kappa, \mu)$-SPACE-FORMS

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ABSTRACT. The object of the present paper is to find some conditions for invariant submanifolds of generalized $(\kappa, \mu)$-space-forms to be totally geodesic.

## 1. Introduction

In 1995, Blair [**1**] introduced the notion of contact metric manifolds with characteristic vector field $\xi$ belonging to the $(\kappa, \mu)$-nullity distribution. Such type of manifolds are called $(\kappa, \mu)$-contact metric manifolds. A contact metric manifold $\bar{M}$ is said to be a generalized $(\kappa, \mu)$-contact metric manifold [**2**] if its curvature tensor $\bar{R}$ satisfies the condition

$$\bar{R}(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\} \tag{1}$$

for some smooth functions $\kappa$ and $\mu$ on $\bar{M}$, independent of choice of vector fields $X$ and $Y$. If $\kappa$ and $\mu$ are constants, then the manifold is called a $(\kappa, \mu)$-contact metric manifold.

A $(\kappa, \mu)$-contact metric manifold $\bar{M}$ of dimension greater than three with constant $\phi$-sectional curvature $c$ is called $(\kappa, \mu)$-space-form [**10**], and the curvature tensor $\bar{R}$ of such a manifold is given by [**10**],

$$\begin{aligned}\bar{R}(X,Y)Z = {} & \Big(\frac{c+3}{4}\Big)R_1(X,Y)Z + \Big(\frac{c-1}{4}\Big)R_2(X,Y)Z \\ & + \Big(\frac{c+3}{4} - k\Big)R_3(X,Y)Z + R_4(X,Y)Z + \frac{1}{2}R_5(X,Y)Z \\ & + (1-\mu)R_6(X,Y)Z,\end{aligned} \tag{2}$$

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where $R_1$, $R_2$, $R_3$, $R_4$, $R_5$, and $R_6$ are defined by

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y,$$
$$R_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z,$$
$$R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(X)\xi - g(Y,Z)\eta(Y)\xi,$$
$$R_4(X,Y)Z = g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y,$$
$$R_5(X,Y)Z = g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX,$$
$$R_6(X,Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi$$

for any vector fields $X, Y \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the Lie algebra of all vector fields on $\bar{M}$, where $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $\mathcal{L}$ is the usual Lie derivative. In [**3**], the authors introduced and studied the notion of generalized $(\kappa,\mu)$-space-forms with several examples. An almost contact metric manifold $(\bar{M}^{2n+1},\phi,\xi,\eta,g)$ is called generalized $(\kappa,\mu)$-space-form if there exists $f_1, f_2, f_3, f_4, f_5, f_6 \in C^\infty(\bar{M})$, the set of smooth functions on $\bar{M}$, such that

$$\begin{aligned}\bar{R}(X,Y)Z = {} & f_1R_1(X,Y)Z + f_2R_2(X,Y)Z + f_3R_3(X,Y)Z,\\ & + f_4R_4(X,Y)Z + f_5R_5(X,Y)Z + f_6R_6(X,Y)Z,\end{aligned} \tag{3}$$

where $R_1, R_2, R_3, R_4, R_5$, and $R_6$ are defined in (2). This manifold of dimension $(2n+1)$ is denoted by $\bar{M}(f_1, f_2, f_3, f_4, f_5, f_6)$.

It is obvious that $(\kappa,\mu)$-space-forms are natural examples of generalized $(\kappa,\mu)$-space -forms, with constant functions

$$f_1 = \frac{c+3}{4}, \qquad f_2 = \frac{c-1}{4}, \qquad f_3 = \frac{c+3}{4} - \kappa$$
$$f_4 = 1, \qquad f_5 = \frac{1}{2}, \qquad f_6 = 1-\mu.$$

A submanifold of an almost contact metric manifold is called invariant if the structure tensor field $\phi$ maps tangent vector fields to tangent vector fields. It is called anti invariant if $\phi$ maps tangent vector fields to normal vector fields. A submanifold is totally geodesic if its second fundamental form vanishes identically. The totally geodesic submanifolds are simplest submanifolds. So their is a natural trend to verify whether invariant or anti invariant submanifolds are totally geodesic. There are so many works in this line, for example, we refer [**4**], [**5**], [**8**], [**14**], [**17**], [**18**].

Totally umbilical submanifolds of almost contact manifolds have been studied in the papers [**6**], [**7**], [**9**], [**11**], [**15**].

In [**6**], the authors characterized totally umbilical submanifolds of Sasakian manifolds using theory of differential equations [**6**], [**12**], [**13**].

Keeping in mind the above works, in this paper, we would like to search the cases when an invariant submanifold of a generalized $(\kappa,\mu)$-space-form is totally geodesic. The same properties is also studied for totally umbilical submanifolds.

The present paper is organized as follows.

After the introduction and preliminaries, we study invariant submanifolds of generalized $(\kappa,\mu)$-space-forms in Section 3. In this section, we have shown that an invariant submanifold of generalized $(\kappa, \mu)$-space-form whose second fundamental form satisfies some specific property is totally geodesic. In the last section, we investigate totally umbilical submanifolds of generalized $(\kappa, \mu)$-space-forms.

## 2. Preliminaries

Let $\bar{M}$ be a $(2n+1)$-dimensional smooth differential manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, g)$, where $\phi$ is a (1,1)-tensor field, $\xi$ is a vector field, $\eta$ is a one form, and $g$ is a compatible Riemannian metric on $\bar{M}$. For such manifolds, we know [**1**]

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \tag{4}$$

$$\eta(X) = g(X, \xi), \tag{5}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{6}$$

$$\phi\xi = 0, \qquad \eta o \phi = 0, \qquad g(X, \phi Y) = -g(\phi X, Y) \tag{7}$$

for any $X, Y \in \Gamma(\bar{M})$, where $\Gamma(\bar{M})$ denotes the Lie algebra of all vector fields on $\bar{M}$.

Given a contact metric manifold $(\bar{M}^{2n+1}, \phi, \xi, \eta, g)$, we define a $(1,1)$ tensor field $h$ by $h = \frac{1}{2}\mathcal{L}_\xi \phi$, where $\mathcal{L}$ is the usual Lie derivative. Then $h$ is symmetric and satisfies the following relations

$$h\xi = 0, \qquad h\phi = -\phi h, \qquad \mathrm{tr}(h) = \mathrm{tr}(\phi h) = 0, \qquad \eta(hX) = 0 \tag{8}$$

for any $X, Y \in \Gamma(\bar{M})$.

Moreover, if $\bar{\nabla}$ denotes the covariant derivative with respect to $g$, then the following relation holds

$$\bar{\nabla}_X \xi = -\phi X - \phi h X. \tag{9}$$

On a generalized $(\kappa, \mu)$-space-form, we also have [**3**]

$$\begin{aligned} \bar{R}(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} \\ + (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY\}, \end{aligned} \tag{10}$$

$$\begin{aligned} \bar{R}(\xi,Y)Z = (f_1 - f_3)\{g(Y,Z)\xi - \eta(Z)Y\} \\ + (f_4 - f_6)\{g(hY,Z)\xi - \eta(Z)hY\}, \end{aligned} \tag{11}$$

$$\bar{S}(\xi,\xi) = 2n(f_1 - f_3), \tag{12}$$

$$\begin{aligned} \eta(\bar{R}(X,Y)Z) = (f_1 - f_3)\{g(Y,Z)\eta(X) - G(X,Z)\eta(Y)\} \\ + (f_4 - f_6)\{g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)\}, \end{aligned} \tag{13}$$

$$\begin{aligned} \bar{S}(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y) \\ + \{(2n-1)f_4 - f_6\}g(hX,Y), \end{aligned} \tag{14}$$

$$\bar{r} = 2n\{(n+1)f_1 + 3f_2 - 2nf_3\}, \tag{15}$$

$$\begin{aligned}\bar{Q}(X) = (2nf_1 + 3f_2 - f_3)X - \{3f_2 + (2n-1)f_3\}\eta(X)\xi \\ + \{(2n-1)f_4 - f_6\}hX\end{aligned} \tag{16}$$

for any $X, Y \in \Gamma(TM)$, where $\bar{R}$, $\bar{S}$, $\bar{r}$, and $\bar{Q}$, are curvature tensor, Ricci tensor, scalar curvature, and Ricci operator on $\bar{M}$, respectively.

In a K-contact manifold, we have [**1**]

$$(\bar{\nabla}_X \phi)(Y) = \bar{R}(\xi, X)Y \tag{17}$$

for any $X, Y \in \Gamma(\bar{M})$. Using (11) and (17), we have in a generalized $(\kappa, \mu)$-space-form $\bar{M}(f_1, f_2, f_3, f_4, f_5, f_6)$,

$$\begin{aligned}(\bar{\nabla}_X \phi)(Y) = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X] \\ + (f_4 - f_6)[g(hX,Y)\xi - \eta(Y)hX],\end{aligned} \tag{18}$$

also, from (18), we get

$$\bar{\nabla}_X \xi = -(f_1 - f_3)\phi X - (f_4 - f_6)\phi hX. \tag{19}$$

Let $M^{2m+1}$ $(m < n)$ be the submanifold of a contact metric manifold $\bar{M}^{2n+1}$. Let $\nabla$ and $\bar{\nabla}$ are the Levi-Civita connections of $M$ and $\bar{M}$, respectively. Then for any vector fields $X, Y \in \Gamma(TM)$, the second fundamental form $\sigma$ is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y). \tag{20}$$

A submanifold of a generalized $(\kappa, \mu)$-space-form is called totally geodesic if

$$\sigma(X, Y) = 0 \qquad \text{for } X, Y \in \Gamma(TM).$$

Furthermore, for any section $N$ of normal bundle $T^{\perp}M$, we have

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp} N \tag{21}$$

where $\nabla^{\perp}$ denotes the normal bundle connection of $M$. The second fundamental form $\sigma$ and shape operator $A_N$ are related by

$$g(A_N X, Y) = g(\sigma(X, Y), N). \tag{22}$$

On a Riemannian manifold $\bar{M}$, for a $(0, k)$-type tensor field $T(k \geq 1)$ and a $(0, 2)$-type tensor field $E$, by $Q(E, T)$, we denote a $(0, k+2)$-type tensor field ([**19**]) defined as follows:

$$\begin{aligned}Q(E,T)(X_1, X_2, \ldots, X_k; X, Y) = & -T((X \wedge_E Y)X_1, X_2, \ldots, X_n) \\ & - T(X_1, (X \wedge_E Y)X_2, \ldots, X_k) - \ldots \\ & - T(X_1, \ldots, (X \wedge_E Y)X_k),\end{aligned} \tag{23}$$

where $(X \wedge_E Y)Z = E(Y, Z)X - E(X, Z)Y$.

## 3. Invariant submanifolds of generalized $(\kappa,\mu)$-space-forms

Let $M^{2m+1}$ be a submanifold of a generalized $(\kappa,\mu)$-space-form $\bar{M}^{2n+1}(n>m)$ such that the characteristic vector field $\xi$ is tangential to $M$. Generally, a submanifold $M$ is said to be invariant submanifold of $\bar{M}$ if $\phi(TM)\subset TM$. On an invariant submanifold $M$ of $\bar{M}$, it follows that $\xi\in\Gamma(TM)$.

Using (19) and (20), we have

$$\nabla_X\xi+\sigma(X,\xi)=-(f_1-f_3)\phi(X)-(f_4-f_6)\phi(hX).$$

Comparing tangential and normal components, we get

$$\nabla_X\xi=-(f_1-f_3)\phi(X)-(f_4-f_6)\phi(hX) \tag{24}$$

and

$$\sigma(X,\xi)=0 \tag{25}$$

for any vector fields $X\in\Gamma(TM)$.

Now using (18) and (20), we have

$$\begin{aligned}(\nabla_X\phi)Y-\sigma(X,\phi Y)+\phi\sigma(X,Y)=(f_1-f_3)&[g(X,Y)\xi-\eta(Y)X]\\ &+(f_4-f_6)[g(hX,Y)\xi-\eta(Y)hX].\end{aligned}$$

Comparing tangential and normal components, we get

$$\begin{aligned}(\nabla_X\phi)(Y)=(f_1-f_3)&[g(X,Y)\xi-\eta(Y)X]\\ &+(f_4-f_6)[g(hX,Y)\xi-\eta(Y)hX],\end{aligned} \tag{26}$$

and

$$\sigma(X,\phi Y)=\phi\sigma(X,Y) \tag{27}$$

for any vector fields $X,Y\in\Gamma(TM)$.

From (1), comparing tangential and normal components, we get

$$R(X,Y)\xi=\kappa\{\eta(Y)X-\eta(X)Y\}+\mu\{\eta(Y)hX-\eta(X)hY\} \tag{28}$$

and

$$R^{\perp}(X,Y)\xi=0.$$

Thus, we have the following lemma.

**Lemma 3.1.** *An invariant submanifold of a generalized $(\kappa,\mu)$-space-form is a generalized $(\kappa,\mu)$-space-form.*

**Lemma 3.2.** *Let $M$ be a three dimensional invariant submanifold of a generalized $(\kappa,\mu)$-space-form $\bar{M}$, then there exist two differentiable distributions $D$ and $D^{\perp}$ on $M$ such that*

$$TM=D\oplus D^{\perp}\oplus\langle\xi\rangle,\qquad \phi(D)\subset D^{\perp},\qquad \phi(D^{\perp})\subset D.$$

*Proof.* If $M$ is a three dimensional submanifold, then the tangent space $TM$ of $M$ is also three dimensional, so we can write $TM = D^1 \oplus \langle\xi\rangle$. Let $X_1 \in D^1$, so $g(X_1, \phi X_1) = 0$ and $g(\xi, \phi X_1) = 0$. So $\phi X_1$ is orthogonal to $X_1$ and $\xi$. Consequently, it is possible to write $D^1 = D \oplus D^\perp$, where $X_1 \in D \subset D^1$ and $\phi X_1 \in D^\perp \subset D^1$. For $\phi X_1 \in D^\perp$, therefore,

$$TM = D \oplus D^\perp \oplus \langle\xi\rangle.$$

Let $\{X_1, \phi X_1, \xi\}$ be the basis of $TM$. X$\in D$ implies X$= \mu X_1$, so $\phi X = \mu\phi X_1 \in D^\perp$, thus $\phi D \subset D^\perp$. Again $Y \in D^\perp$ implies $Y = \lambda Y_1$, so $\phi Y = \lambda\phi Y_1$, thus $\phi Y = \lambda\phi^2 X_1 = \lambda(-X_1 + \eta(X_1)\xi) = -\lambda X_1 \in D$, therefore, $\phi D \subset D^\perp$. This proves the lemma. □

**Theorem 3.1.** *Every three dimensional invariant submanifold of a generalized $(\kappa, \mu)$-space-form is totally geodesic.*

*Proof.* Let $M$ be a three dimensional manifold of a generalized $(\kappa, \mu)$-space-form $\bar{M}$. Let $X_1, Y_1 \in D$, and consequently $\phi X_1, \phi Y_1 \in D^\perp$. In view of (4) and (27), we obtain

$$\begin{aligned}\sigma(\phi X_1, \phi Y_1) &= \phi^2\sigma(X_1, Y_1) = -\sigma(X_1, Y_1) + \eta(\sigma(X_1, Y_1))\xi \\ &= -\sigma(X_1, Y_1).\end{aligned}$$

Let $\phi X_1 = X_2$, $\phi Y_1 = Y_2$. We note that $X_2 \in D^\perp$ and $Y_2 \in D^\perp$. Therefore,

$$\sigma(X_2, Y_2) = -\sigma(X_1, Y_1) \tag{29}$$

for any $X_1, Y_1 \in D$ and $X_2, Y_2 \in D^\perp$. Since $\sigma$ is bilinear,

$$\sigma(X_1 + X_2 + \xi, Y_1) = \sigma(X_1, Y_1) + \sigma(X_2, Y_1) + \sigma(\xi, Y_1), \tag{30}$$

$$\sigma(X_1 + X_2 + \xi, Y_2) = \sigma(X_1, Y_2) + \sigma(X_2, Y_2) + \sigma(\xi, Y_2), \tag{31}$$

$$\sigma(X_1 + X_2 + \xi, \xi) = \sigma(X_1, \xi) + \sigma(X_2, \xi) + \sigma(\xi, \xi). \tag{32}$$

Keeping in mind $\sigma(X, \xi) = 0$ for $X \in TM$, and using (30), (31), (32), we get

$$\sigma(X_1 + X_2 + \xi, Y_1 + Y_2 + \xi) = \sigma(X_1, Y_2) + \sigma(X_2, Y_1) \tag{33}$$

and

$$\sigma(X_1 + X_2 + \xi, Y_1 - Y_2 + \xi) = \sigma(X_1, Y_2) - \sigma(X_2, Y_1). \tag{34}$$

Now since

$$TM = D \oplus D^\perp \oplus \langle\xi\rangle,$$

any arbitrary vector fields $U, V$ of $TM$ can be taken as $U = X_1 + X_2 + \xi$ and $V = Y_1 - Y_2 + \xi$. Then from equation (34), we have

$$\sigma(U, V) = \sigma(X_2, Y_1) - \sigma(X_1, Y_2) = \sigma(\phi X_1, Y_1) - \sigma(X_1, \phi Y_1) = 0.$$

Hence the submanifold $M$ is totally geodesic. □

**Example 3.1.** In the following, we give an example of invariant submanifold of a generalized $(\kappa,\mu)$-space-form. The example is taken from [**16**]. We give it here for illustration. Let us consider the five dimensional manifold $\bar{M} = \{(x_1,x_2,x_3,x_4,t) \in \mathbb{R}^5 : t \neq 0\}$, where $(x_1,x_2,x_3,x_4,t)$ are the standard coordinates of $\mathbb{R}^5$. We choose the vector fields

$$e_1 = \mathrm{e}^{-t}\frac{\partial}{\partial x_1}, \quad e_2 = \mathrm{e}^{-t}\frac{\partial}{\partial x_2}, \quad e_3 = \mathrm{e}^{-t}\frac{\partial}{\partial x_3}, \quad e_4 = \mathrm{e}^{-t}\frac{\partial}{\partial x_4}, \quad e_5 = \frac{\partial}{\partial t},$$

which are linearly independent at each point of $\bar{M}$.
We define $g$ such that $\{e_1,e_2,e_3,e_4,e_5\}$ are orthonormal basis of $\bar{M}$, i.e.,

$$g(e_i,e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \qquad \text{where } 1 \leq i,j \leq 5. \end{cases}$$

We consider a 1-form $\eta$ defined by

$$\eta(X) = g(X,e_5), \qquad X \in \Gamma(T\bar{M}),$$

i.e., we choose $e_5 = \xi$. We define the $(1,1)$ tensor field $\phi$ by

$$\phi(e_1) = e_3, \quad \phi(e_2) = e_4, \quad \phi(e_3) = -e_1, \quad \phi(e_4) = -e_2, \quad \phi(e_5) = 0.$$

The linear property of $g$ and $\phi$ shows that

$$\eta(e_5) = 1, \quad \phi^2(X) = -X + \eta(X)e_5,$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for $X, Y \in \Gamma(T\bar{M})$. Hence $\bar{M}(\phi,\xi,\eta,g)$ defines an almost contact manifold with $e_5 = \xi$. Moreover, let $\bar{\nabla}$ be the Levi-Civita connection with respect to metric $g$. Then, we have

$$[e_i, e_5] = e_i \qquad i = 1,2,3,4,$$
$$[e_i, e_j] = 0, \qquad \text{otherwise}.$$

By Koszul formula, we obtain the following

$$\bar{\nabla}_{e_1}e_1 = -e_5, \quad \bar{\nabla}_{e_2}e_2 = -e_5, \quad \bar{\nabla}_{e_3}e_3 = -e_5, \quad \bar{\nabla}_{e_4}e_4 = -e_5,$$
$$\bar{\nabla}_{e_i}e_j = 0, \quad \text{otherwise}.$$

The tensor field $h$ satisfies

$$he_1 = e_1, \quad he_2 = e_2, \quad he_3 = e_3, \quad he_4 = e_4, \quad he_5 = 0.$$

Now, from the definition of curvature tensor, we obtain

$$R(e_1,e_5)e_1 = e_5, \quad R(e_2,e_5)e_2 = e_5, \quad R(e_3,e_5)e_3 = e_5, \quad R(e_4,e_5)e_4 = e_5,$$
$$R(e_i,e_j)e_k = 0, \quad \text{otherwise}.$$

Thus $\bar{M}$ is a generalized $(\kappa,\mu)$-space-form with $f_1 = -\frac{1}{2}$, $f_2 = 0$, $f_3 = \frac{1}{2}$, $f_4 = 0$, $f_5 = 0$, $f_6 = 0$.

Let M be a subset of $\bar{M}$ and consider the isometric immersion $f\colon M \to \bar{M}$ defined by

$$f(x_1,x_3,t) = (x_1,0,x_3,0,t).$$

It is easy to prove that $M = \{(x_1, x_3, t) \in \mathbb{R}^3 : t \neq 0\}$ is a submanifold of $\bar{M}$, where $(x_1, x_3, t)$ are the standard co-ordinate of $\mathbb{R}^3$. We choose the vector fields

$$e_1 = \mathrm{e}^{-t}\frac{\partial}{\partial x_1}, \qquad e_3 = \mathrm{e}^{-t}\frac{\partial}{\partial x_3}, \qquad e_5 = \frac{\partial}{\partial t},$$

which are linearly independent at each point of $M$. We define $g_1$ such that $\{e_1, e_3, e_5\}$ are orthonormal basis of $M$, i.e.,

$$g_1(e_i, e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \quad \text{where } i = 1, 3, 5. \end{cases}$$

We define a 1-form $\eta_1$ and a $(1,1)$ tensor $\phi_1$, respectively, by

$$\eta_1 = g_1(X, e_5),$$

and

$$\phi_1(e_1) = e_3, \qquad \phi_1(e_3) = -e_1, \qquad \phi_1(e_5) = 0.$$

The linear property of $g_1$ and $\phi_1$ shows that

$$\eta_1(e_5) = 1, \qquad \phi_1^2(X) = -X + \eta_1(X)e_5,$$
$$g_1(\phi_1 X, \phi_1 Y) = g_1(X, Y) - \eta_1(X)\eta_1(Y)$$

for $X, Y \in \Gamma(TM)$. Hence $M(\phi_1, \xi, \eta_1, g_1)$ is an invariant submanifold of $\bar{M}$ with $e_5 = \xi$. Moreover, let $\nabla$ be the Levi-Civita connection with respect to the metric $g_1$. Then, we have

$$[e_i, e_5] = e_i, \qquad \text{for } i, j = 1, 3,$$
$$[e_i, e_j] = 0, \qquad \text{otherwise.}$$

By using Kouszul formula, we obtain

$$\nabla_{e_1} e_1 = -e_5, \qquad \nabla_{e_3} e_3 = -e_5, \qquad \nabla_{e_5} e_5 = 0,$$
$$\nabla_{e_1} e_5 = e_1, \qquad \nabla_{e_3} e_5 = e_3, \qquad \nabla_{e_5} e_1 = 0,$$
$$\nabla_{e_5} e_3 = 0, \qquad \nabla_{e_1} e_3 = 0, \qquad \nabla_{e_3} e_1 = 0.$$

Using the above results, we see that $\sigma(X, Y) = 0$. So the submanifold is totally geodesic.

Hence the Theorem 3.1 is verified.

We have the following Lemma of [**3**],

**Lemma 3.3.** *If $\bar{M}$ is a $(\kappa, \mu)$-space-form, then $k \leq 1$. If $k = 1$, then $h = 0$ and $M$ is a Sasakian manifold. If $\kappa < 1$, then $M$ admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$, and $D(-\lambda)$ determined by eigenspaces of $h$, where $\lambda = \sqrt{1-k}$. Moreover, if $X \in D(\lambda)$, then $hX = \lambda X$, and if $X \in D(-\lambda)$, then $hX = -\lambda X$.*

**Theorem 3.2.** *An invariant submanifold of a generalized $(\kappa, \mu)$-space-form is totally geodesic if and only if $Q(S, \bar{\nabla}\sigma) = 0$, provided $2n(f_1 - f_3)[(f_1 - f_3) \pm (f_4 - f_6)\lambda] \neq 0$.*

*Proof.* Assume $Q(S, \bar{\nabla}\sigma) = 0$, then

$$Q(S, \bar{\nabla}_X \sigma)(W, K; U, V) = 0$$

for the vector fields $X, W, K, U, V \in \Gamma(TM)$. By the above equation and (23), we have

$$\begin{aligned} 0 = & -(\bar{\nabla}_X\sigma)(S(V,W)U,K) + (\bar{\nabla}_X\sigma)(S(U,W)V,K) \\ & -(\bar{\nabla}_X\sigma)(W,S(V,K)U) + (\bar{\nabla}_X\sigma)(W,S(U,K)V). \\ 0 = & -\nabla^{\perp}_X\sigma(S(V,W)U,K) + \sigma(\nabla_X S(V,W)U,K) + \sigma(S(V,W)U,\nabla_X K) \\ & + \nabla^{\perp}_X\sigma(S(U,W)V,K) - \sigma(\nabla_X S(U,W)V,K) - \sigma(S(U,W)V,\nabla_X K) \\ & - \nabla^{\perp}_X\sigma(W,S(V,K)U) + \sigma(\nabla_X W,S(V,K)U) + \sigma(W,\nabla_X S(V,K)U) \\ & + \nabla^{\perp}_X\sigma(W,S(U,K)V) - \sigma(\nabla_X W,S(U,K)V) - \sigma(W,\nabla_X S(U,K)V). \end{aligned}$$

Using equation (26) and putting $K = V = W = \xi$ in the above equation, we have

$$S(\xi,\xi)\sigma(U,\nabla_X\xi) = 0. \tag{35}$$

By the Lemma 3.3 and the equations (12), (26) and (37), we have

$$2n(f_1 - f_3)[(f_1 - f_3) \pm (f_4 - f_6)\lambda]\sigma(U, \phi X) = 0, \tag{36}$$

according as $X$ in $D(\pm\lambda)$.
By the assumed condition $2n(f_1 - f_3)[(f_1 - f_3) \pm (f_4 - f_6)\lambda] \neq 0$, from above equation, we have

$$\sigma(U, \phi X) = 0.$$

Hence by using (27), we have

$$\sigma(U, X) = 0$$

for any $U, X \in \Gamma(TM)$. Thus the submanifold is totally geodesic.
Converse part is trivially true. This completes the proof. □

**Theorem 3.3.** *An invariant submanifold of a generalized $(\kappa, \mu)$-space-form is totally geodesic if and only if $Q(S, \bar{R}.\sigma) = 0$, provided $2n(f_1 - f_3)[(f_1 - f_3) \pm (f_4 - f_6)\lambda] \neq 0$.*

*Proof.* Assume $Q(S, R.\sigma) = 0$, then

$$Q(S, \bar{R}(X,Y).\sigma)(W, K; U, V) = 0$$

for the vector fields $X, Y, W, K, U, V \in \Gamma(\bar{M})$. From (23), we have

$$\begin{aligned} 0 = & -S(V,W)(\bar{R}(X,Y).\sigma)(U,K) + S(U,W)(\bar{R}(X,Y).\sigma)(V,K) \\ & -S(V,K)(\bar{R}(X,Y).\sigma)(W,U) + S(U,K)(\bar{R}(X,Y).\sigma)(W,V) \\ = & -S(V,W)[R^{\perp}(X,Y)\sigma(U,K) - \sigma(R(X,Y)U,K) - \sigma(R(X,Y)K,U)] \\ & + S(U,W)[R^{\perp}(X,Y)\sigma(V,K) - \sigma(R(X,Y)V,K) - \sigma(R(X,Y)K,V)] \\ & - S(V,K)[R^{\perp}(X,Y)\sigma(W,U) - \sigma(R(X,Y)W,U) - \sigma(R(X,Y)U,W)] \\ & + S(U,K)[R^{\perp}(X,Y)\sigma(W,V) - \sigma(R(X,Y)W,V) - \sigma(R(X,Y)V,W)]. \end{aligned}$$

Using equation (26) and putting $K = V = W = Y = \xi$ in the above equation, we have

$$S(\xi,\xi)\sigma(U,R(X,\xi)\xi) = 0, \tag{37}$$

By the Lemma 3.3 and the equations (11), (26), and (37), we have

$$2n(f_1 - f_3)[(f_1 - f_3) \pm (f_4 - f_6)\lambda]\sigma(U,X) = 0. \tag{38}$$

By the assumed condition $2n(f_1 - f_3)[(f_1 - f_3) \pm (f_4 - f_6)\lambda] \neq 0$, from above equation, we have

$$\sigma(U,X) = 0$$

for any $U, X \in \Gamma(TM)$. Thus the submanifold is totally geodesic.

Converse part is trivially true. This completes the proof. □

**Theorem 3.4.** *An invariant submanifold of a generalized $(\kappa,\mu)$-space-form is totally geodesic if and only if $Q(g,\bar{R}.\sigma) = 0$, provided $[(f_1 - f_3) \pm (f_4 - f_6)\lambda] \neq 0$.*

*Proof.* Assume $Q(g,\bar{R}.\sigma) = 0$, then

$$Q(g,\bar{R}(X,Y).\sigma)(W,K;U,V) = 0$$

for the vector fields $X, Y, W, K, U, V \in \Gamma(TM)$. From (23), we have

$$\begin{aligned} 0 = & -g(V,W)(\bar{R}(X,Y).\sigma)(U,K) + g(U,W)(\bar{R}(X,Y).\sigma)(V,K) \\ & -g(V,K)(\bar{R}(X,Y).\sigma)(W,U) + g(U,K)(\bar{R}(X,Y).\sigma)(W,V) \\ = & -g(V,W)[R^{\perp}(X,Y)\sigma(U,K) - \sigma(R(X,Y)U,K) - \sigma(R(X,Y)K,U)] \\ & + g(U,W)[R^{\perp}(X,Y)\sigma(V,K) - \sigma(R(X,Y)V,K) - \sigma(R(X,Y)K,V)] \\ & - g(V,K)[R^{\perp}(X,Y)\sigma(W,U) - \sigma(R(X,Y)W,U) - \sigma(R(X,Y)U,W)] \\ & + g(U,K)[R^{\perp}(X,Y)\sigma(W,V) - \sigma(R(X,Y)W,V) - \sigma(R(X,Y)V,W)]. \end{aligned}$$

Using equation (26) and putting $K = V = W = Y = \xi$ in the above equation, we have

$$\sigma(R(X,\xi)\xi,U) = 0. \tag{39}$$

By the Lemma 3.3 and the Equations (11), (26), and (39), we have

$$[(f_1 - f_3) \pm (f_4 - f_6)\lambda]\sigma(U,X) = 0. \tag{40}$$

By the assumed condition $[(f_1 - f_3) \pm (f_4 - f_6)\lambda] \neq 0$, from above equation, we have

$$\sigma(U,X) = 0$$

for any $U, X \in \Gamma(TM)$. Thus the submanifold is totally geodesic.

Converse part is trivially true. This completes the proof.

For a $(2n+1)$ dimensional Riemannian manifold $\bar{M}$, the concircular curvature tensor $C$ is defined by [**5**]

$$C(X,Y)Z = \bar{R}(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \tag{41}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$. □

**Theorem 3.5.** *An invariant submanifold of a generalized $(\kappa, \mu)$-space-form is totally geodesic if and only if $Q(g, C.\sigma) = 0$, provided $[(f_1 - f_3) \pm (f_4 - f_6)\lambda - \frac{r}{2n(2n+1)}] \neq 0$.*

*Proof.* Assume $Q(g, C.\sigma) = 0$, then

$$Q(g, C(X, Y).\sigma)(W, K; U, V) = 0$$

for the vector fields $X, Y, W, K, U, V \in \Gamma(TM)$. From (23), we have

$$\begin{aligned} 0 = & -g(V, W)(C(X, Y).\sigma)(U, K) + g(U, W)(C(X, Y).\sigma)(V, K) \\ & - g(V, K)(C(X, Y).\sigma)(W, U) + g(U, K)(C(X, Y).\sigma)(W, V) \\ = & -g(V, W)[C^{\perp}(X, Y)\sigma(U, K) - \sigma(C(X, Y)U, K) - \sigma(C(X, Y)K, U)] \\ & + g(U, W)[C^{\perp}(X, Y)\sigma(V, K) - \sigma(C(X, Y)V, K) - \sigma(C(X, Y)K, V)] \\ & - g(V, K)[C^{\perp}(X, Y)\sigma(W, U) - \sigma(C(X, Y)W, U) - \sigma(C(X, Y)U, W)] \\ & + g(U, K)[C^{\perp}(X, Y)\sigma(W, V) - \sigma(C(X, Y)W, V) - \sigma(C(X, Y)V, W)]. \end{aligned}$$

Using equation (26) and putting $K = V = W = Y = \xi$ in the above equation, we have

$$\sigma(C(X, \xi)\xi, U) = 0. \tag{42}$$

By the Lemma 3.3 and the equations (11), (26), (41), and (42), we have

$$[(f_1 - f_3) \pm (f_4 - f_6)\lambda - \frac{r}{2n(2n+1)}]\sigma(U, X) = 0. \tag{43}$$

By the assumed condition $[(f_1 - f_3) \pm (f_4 - f_6)\lambda - \frac{r}{2n(2n+1)}] \neq 0$, from above equation, we have

$$\sigma(U, X) = 0$$

for any $U, X \in \Gamma(TM)$. Thus the submanifold is totally geodesic.

Converse part is trivially true. This completes the proof. □

**Theorem 3.6.** *An invariant submanifold of a generalized $(\kappa, \mu)$-space-form is totally geodesic if and only if $Q(S, C.\sigma) = 0$, provided $2n(f_1 - f_3)[(f_1 - f_3) \pm (f_4 - f_6)\lambda - \frac{r}{2n(2n+1)}] \neq 0$.*

*Proof.* Assume $Q(S, C.\sigma) = 0$, then

$$Q(S, C(X, Y).\sigma)(W, K; U, V) = 0$$

for the vector fields $X, Y, W, K, U, V \in \Gamma(TM)$. From (23), we have

$$\begin{aligned} 0 = & -S(V, W)(C(X, Y).\sigma)(U, K) + S(U, W)(C(X, Y).\sigma)(V, K) \\ & - S(V, K)(C(X, Y).\sigma)(W, U) + S(U, K)(C(X, Y).\sigma)(W, V) \\ = & -S(V, W)[C^{\perp}(X, Y)\sigma(U, K) - \sigma(C(X, Y)U, K) - \sigma(C(X, Y)K, U)] \\ & + S(U, W)[C^{\perp}(X, Y)\sigma(V, K) - \sigma(C(X, Y)V, K) - \sigma(C(X, Y)K, V)] \\ & - S(V, K)[C^{\perp}(X, Y)\sigma(W, U) - \sigma(C(X, Y)W, U) - \sigma(C(X, Y)U, W)] \\ & + S(U, K)[C^{\perp}(X, Y)\sigma(W, V) - \sigma(C(X, Y)W, V) - \sigma(C(X, Y)V, W)]. \end{aligned}$$

Using equation (26) and putting $K = V = W = Y = \xi$ in the above equation, we have

$$S(\xi,\xi)\sigma(U, C(X,\xi)\xi) = 0. \tag{44}$$

By the Lemma 3.3 and the equations (11), (26), (41), and (44), we have

$$2n(f_1 - f_3)\Big[(f_1 - f_3) \pm (f_4 - f_6)\lambda - \frac{r}{2n(2n+1)}\Big]\sigma(U, X) = 0. \tag{45}$$

By the assumed condition $2n(f_1 - f_3)\big[(f_1 - f_3) \pm (f_4 - f_6)\lambda - -\frac{r}{2n(2n+1)}\big] \neq 0$, from above equation, we have

$$\sigma(U, X) = 0$$

for any $U, X \in \Gamma(TM)$. Thus the submanifold is totally geodesic.

Converse part is trivially true. This completes the proof. □

**Remark.** The Theorems 3.5 and 3.6 have analogue for projective curvature tensor, conformal curvature tensor, and conharmonic curvature tensor. The proofs are similar.

## 4. Totally Umbilical submanifolds of a generalized $(\kappa, \mu)$-space-forms

Let $M$ be a totally umbilical submanifold of a generalized $(\kappa, \mu)$-space-form $\bar{M}$. Then the second fundamental form $\sigma$ of $M$ is given by $\sigma(X, Y) = g(X, Y)H$ [**6**], where $X, Y \in \Gamma(TM)$ and $H$ is mean curvature vector.

If we set $\alpha = \|H\|^2$, then for the totally umbilical submanifold $M$ with mean curvature parallel in the normal bundle, we have $X.\alpha = 0$ for any $X \in \Gamma(TM)$, that is, $\alpha$ is constant.

If $\alpha \neq 0$, define a unit vector $e \in \nu$ in the normal bundle, by setting $H = \sqrt{\alpha}e$. The normal bundle can be split into the direct sum $\alpha = \{e\} \oplus \{e\}^{\perp}$, where $\{e\}^{\perp}$ is the orthogonal compliment of the line sub-bundle $\{e\}$ spanned by $e$. For each $X \in \Gamma(TM)$, set

$$\phi X = \psi(X) - A(X)e + P(X), \qquad \phi e = t + F, \tag{46}$$

where $\psi(x)$ is the tangential components of $\phi X$, while $A(X)$ and $P(X)$ are the $\{e\}$ and $\{e\}^{\perp}$ components, respectively. $t$ and $F$ are the $\{e\}$ and $\{e\}^{\perp}$ components of $\phi e$, respectively, in view of the skew-symmetry of $\phi$.

**Lemma 4.1.** *Let $M$ be a totally umbilical submanifold of a generalized $(\kappa, \mu)$-space-form $\bar{M}$ with curvature vector parallel to the normal bundle. If $\mu \neq 0$, then for any $X \in \Gamma(TM)$, following equations hold:*

i) $\bar{\nabla}_X e = -\sqrt{\alpha}X$,
ii) $\nabla_X t = -\sqrt{\alpha}\psi(X) - [f_1 - f_3 \pm \lambda(f_4 - f_6)]\eta(e)X$,
iii) $\nabla^{\perp}_X F = -\sqrt{\alpha}P(X)$.

*Proof.* Taking inner product with respect to $Y$ in both sides of equation (21), we obtain

$$\bar{\nabla}_X N = -g(H,N)X + \nabla^{\perp}_X N.$$

Putting $N = e$ in above equation, we obtain

$$\bar{\nabla}_X e = -\sqrt{\alpha}X.$$

Thus (i) is proved.

Next put $Y = e$ in the equation (18), and using the Lemma 3.3 and the equation (46), we obtain

$$\begin{aligned}&\nabla_X t + \nabla^{\perp}_X F + \sqrt{\alpha}(\psi(X) - A(X)e + P(X)) + \sigma(X,\phi(e))\\ &= -[f_1 - f_3 \pm \lambda(f_4 - f_6)]\eta(e)X\end{aligned}$$

Next comparing the tangential part, we have

$$\nabla_X t = -\sqrt{\alpha}\psi(X) - [f_1 - f_3 \pm \lambda(f_4 - f_6)]\eta(e)X.$$

Thus (ii) is proved. Now comparing $\{e\}^{\perp}$ component and using the result $A(X) = g(X,t)$, we obtain

$$\nabla^{\perp}_X F = -\sqrt{\alpha}P(X).$$

Thus (iii) is proved. □

**Lemma 4.2.** *Let $M$ be a totally umbilical submanifold of a generalized $(\kappa,\mu)$-space-form $\bar{M}$ with mean curvature vector parallel in the normal bundle. If $\mu \neq 0$ and $\xi \perp e$, then setting $\xi = \xi_1 + \xi_2$, where $\xi_1$ is the tangential component and $\xi_2$ is the $\{e\}^{\perp}$-component of $\xi$, we have*

(i) $\nabla_X \xi_1 = -[(f_1 - f_3) \pm \lambda(f_4 - f_6)]\psi(X)$,

(ii) $(\nabla_X\psi)Y = [(f_1 - f_3) \pm \lambda(f_4 - f_6) - \frac{\alpha}{(f_1-f_3)\pm\lambda(f_4-f_6)}](g(X,Y)\xi_1 - \eta(Y)X)$.

*Proof.* Putting $\xi = \xi_1 + \xi_2$ in the equation (2.15) and (4.1), we have

$$\nabla_X\xi_1 + \nabla_X\xi_2 + \sigma(X,\xi) = ([(f_1 - f_3) \pm \lambda(f_4 - f_6)]\psi X - A(X)e + P(X)).$$

Comparing tangential part, we have (i), and comparing e component, we have $\sigma(X,\xi) = [(f_1 - f_3) \pm \lambda(f_4 - f_6)]A(X)e$, i.e.,

$$\sqrt{\alpha}\eta(X) = [(f_1-f_3)\pm\lambda(f_4-f_6)]A(X), \qquad \sqrt{\alpha}\xi_1 = [(f_1-f_3)\pm\lambda(f_4-f_6)]t. \tag{47}$$

Now using the equations (18) and (46), we have

$$\begin{aligned}&\nabla_X(\psi Y) - \nabla_X(AY)e - A(Y)(\nabla_X e) - \psi(\nabla_X Y) + A(\nabla_X Y)e\\ &\quad - P(\nabla_X Y) + (\nabla_X PY)\\ &= [(f_1 - f_3) \pm \lambda(f_4 - f_6)](g(X,Y)\xi - \eta(Y)X).\end{aligned}$$

Using the Lemma 4.1, from the above equation, we have

$$\begin{aligned}&(\nabla_X\psi)Y + (\nabla_X P)Y + \sqrt{\alpha}g(X,Y)(t+F) - \frac{\sqrt{\alpha}\eta(Y)X}{[(f_1 - f_3) \pm \lambda(f_4 - f_6)]}\\ &= [(f_1 - f_3) \pm \lambda(f_4 - f_6)](g(X,Y)\xi - \eta(Y)X).\end{aligned}$$

Comparing the tangential part, we obtain (ii). □

**Theorem 4.1.** *Let $M$ be an $n$ dimensional totally umbilical submanifold of a generalized $(\kappa,\mu)$-space-form with mean curvature vector parallel in the normal bundle. Then one of the following holds:*

(i) *$M$ is totally geodesic,*
(ii) *$M$ is isometric to a sphere,*
(iii) *$M$ is homothetic to a Sasakian manifold.*

*Proof.* Since $H$ is parallel in the normal bundle, $\mu$ is a constant. If $\mu = 0$, then $H = 0$, and consequently $\sigma(X,Y) = 0$, $X, Y \in \Gamma(TM)$. Thus the submanifold $M$ is totally geodesic, which proves the first part of the theorem.

Next, we assume that $\mu \neq 0$. Define a smooth function $f\colon M \to R$ by $f = g(e,\xi), X \in \Gamma(TM)$. Then Lemma 4.1, and equations (19), (20), (21) imply that

$$\begin{aligned} Xf &= g(\nabla_X \xi, e) + g(\xi, \nabla_X e) \\ &= [(f_1 - f_3) \pm \lambda(f_4 - f_6)]g(X,t) - \sqrt{\alpha} g(\xi, X). \end{aligned}$$

So, by using the equations (19), (20), (21), (46), and the Lemma 4.2, we have

$$XYf - (\nabla_X Y)f = -[(f_1 - f_3) \pm \lambda(f_4 - f_6)]^2 f g(X,Y),$$

$$g(\nabla_X \mathrm{grad} f, Y) = -[(f_1 - f_3) \pm \lambda(f_4 - f_6)]^2 f g(X,Y). \tag{48}$$

Taking trace of this equation, we have

$$\Delta f = -[(f_1 - f_3) \pm \lambda(f_4 - f_6)]^2 n f. \tag{49}$$

Then, if $f$ is a non-constant function, then the equation (49) is the differential equation in [**11**], which is necessary and sufficient condition for $M$ to be isometric to a sphere of radius $\frac{1}{(f_1 - f_3) \pm \lambda(f_4 - f_6)}$.

If $f$ is a constant, then Equation (49) gives $-n[(f_1 - f_3) \pm \lambda(f_4 - f_6)]^2 f = 0$, $\alpha$ is non-zero, and consequently $f = 0$, that is, $\xi \perp e$.

Now define a smooth function $G\colon M \to R$ by

$$G = \frac{1}{2} tr.\psi^2. \tag{50}$$

Note that (46) gives $g(\psi Y, X) = -g(\psi X, Y)$, $X, Y \in \Gamma(TM)$.

Let $\omega$ be a 1-form defined by $\omega = dG$. For each $p \in M$, we can choose a local orthonormal frame $\{e_1, \dots, e_n\}$ of $M$ such that $\nabla e_i(p) = 0$. Thus, for any $Z \in \Gamma(TM)$, we have

$$\omega(Z) = ZG = \sum_{i=1}^{n} g((\nabla_Z \psi)(e_i), \psi(e_i)). \tag{51}$$

Using the Lemma 4.2, we obtain

$$\omega(Z) = 2Ng(\psi Z, \xi_1), \tag{52}$$

where $N = [(f_1 - f_3) \pm \lambda(f_4 - f_6) - \frac{\alpha}{(f_1 - f_3) \pm \lambda(f_4 - f_6)}]$.

The first covariant derivative of (52) is

$$(\nabla\omega)(Y,Z) = 2N([(f_1 - f_3) \pm \lambda(f_4 - f_6)]g(\psi Y, \psi Z)) + 2N^2[g(\xi_1, \xi_1)g(Y,Z) - g(Y,\xi)g(Z,\xi_1)].$$

And consequently using the equation (4.7) and the above equation, we have

$$(\nabla^2\omega)(X,Y,Z) + N[(f_1 - f_3) \pm \lambda(f_4 - f_6)](2g(Y,Z)\omega(X) + g(X,Y)\omega(Z) + g(X,Z)\omega(Y)) = 0. \tag{53}$$

Equation (53) is the differential equation in [**7**] which $G$ being non-constant, is the necessary and sufficient condition for $M$ to be isometric to a sphere. This again leads to case (ii). Suppose $G$ is a constant function. Then Equation (53) gives $\psi(\xi_1) = 0$. Define a smooth function $G_1 \colon M \to R$ by

$$G_1 = g(\xi_1, \xi_1).$$

Then using the Lemma 4.2, we get $X\alpha = 0$, $X \in \Gamma(TM)$. In others words, $\xi_1$ has a constant length. Taking the covariant derivative in (i) of Lemma 4.2 and using (ii), we get

$$\nabla_X\nabla_Y\xi_1 - \nabla_{\nabla_X Y}\xi_1 = N[(f_1 - f_3) \pm \lambda(f_4 - f_6)](g(X,Y)\xi_1 - g(Y,\xi_1)X). \tag{54}$$

Further more, from (i) of the Lemma 4.2, it follows that $\xi_1$ is a Killing vector field. Since $N \neq 0$ as $[(f_1 - f_3) \pm \lambda(f_4 - f_6) - \frac{\alpha}{(f_1-f_3)\pm\lambda(f_4-f_6)}] \neq 0$ and $\xi_1$ is a Killing vector field of a constant length which satisfies (54) A result of Okumura [**19**] states that if $\xi_1 \neq 0$, then $M$ is homothetic to a Sasakian manifold, which is (iii). Thus to complete the proof, we have only to show that $\xi_1 = 0$ cannot happen.

We see that if $\xi_1 = 0$, then $\xi \in \{e\}^\perp$ since $\xi \perp e$. Lemma 4.2 gives $\psi(X) = 0$, thus $\phi X$ is normal to $M$ for all $X \in \Gamma(TM)$. Again, equation (47) gives $t = 0$, consequently $\phi e = F \in \{e\}^\perp$, and $g(\phi X, \phi e) = g(X,e) - \eta(X)\eta(e) = 0$, $X \in \Gamma(TM)$, $g(\phi e, \xi) = 0$. Thus the dim of

$$\nu \geq \dim\{M\} + \dim\{\xi\} + \dim\{e\} + \dim\{\phi e\} - 1,$$

which is impossible as $\dim\{\bar{M}\} = 2n+1$. This completes the proof. □

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