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EDGE COLORINGS AVOIDING PATTERNS

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Abstract. We say that a pattern is a graph together with an edge coloring, and a pattern $P = (H, c)$ occurs in some edge coloring c' of G if c', restricted to some subgraph of G isomorphic to H, is equal to c up to renaming the colors. Inspired by Matoušek's visibility blocking problem, we study edge colorings of cliques that avoid certain patterns.

We show that for every pattern P, such that the number of edges in P is at least the number of vertices in P plus the number of colors minus 2, there is an edge coloring of K_n that avoids P and uses linear number of colors; the same also holds for finite sets of such patterns.

1. Introduction

This paper is a result an unsuccessful attempt to solve the visibility blocking problem formulated by Matoušek **[10]**, that itself originated from considerations on the big-line-big-clique conjecture **[8]**. Consider a set $V \subseteq \mathbb{R}^2$ of finite order n, such that no three points from V are collinear. A *visibility-blocking set for V* is a set C disjoint from V such that every segment with endpoints in V contains at least one point from C. By $b(V)$ we denote the minimum cardinality of a visibility-blocking set for V and $b(n)$ denotes the minimum of $b(V)$ over all sets $V \subset \mathbb{R}^2$ of order n such that no three points in V are collinear.

We know that $b(n)$ is at least $\left(\frac{25}{8} - o(1)\right) n$ **[3]**. On the other hand, the best known upper bound is $ne^{c\sqrt{\log n}}$ for some constant c – it follows from a result regarding a related problem, where the blocking points are required to be middle points of the segments **[11]**.

Is $b(n)$ linear or superlinear? Some clues seem to suggest the latter. If we add an extra constraint that points in V should be in a convex position, then the lower bound increases to $\Theta\left(n \log n\right)$ **[9, 10]**. The answer is also superlinear in the middle points variant **[11]**.

We look at this problem as edge-coloring of a complete graph. Let V be a finite subset of $\mathbb{R}^2$ of order n and C a visibility-blocking set for V. We define an edge coloring of K_n in the following way. Suppose that vertices of K_n are identified with points from V. A color of an edge uv is a point from C that lies on the segment with endpoints u and v (if there are many such points from C, we pick

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one arbitrarily). Any edge-coloring of a clique that can be obtained in this way will be called a *visibility-blocking coloring*. Note that $b(n)$ is the minimum number of colors in a visibility-blocking coloring of K_n.

Ideally, we would like to work on a combinatorial characterization of visibility-blocking colorings, but we do not have it. However, we can name a few properties, starting with the simplest one: a visibility-blocking coloring is proper (note that if there were edges uv and uw of the same color c, then the line containing points u and c would also contain v and w, which contradicts the assumption that no three points from V are collinear). Clearly $\chi'(K_n) \geq n-1$, which implies a rather weak result that $b(n) \geq n-1$.

Visibility-blocking colorings also do not contain 2-colored 4-cycles, i.e. vertices u, v, w, x such that pairs of edges (uv, wx) and (ux, vw) that share a color. Indeed, if $c(uw) = c(vx)$, then points v and x are on different sides on the line containing u and w, so intervals ux and vw can not intersect, hence $c(ux) \neq c(vw)$. Even more restrictive colorings have already been studied and are called acyclic edge colorings (they are proper edge colorings with no 2-colored cycles). it is possible to find an acyclic edge-coloring of a graph of maximum degree Δ with the number of colors linear in Δ – Alon, McDiarmid and Reed proved an upper bound 64Δ [**1**], but later it was improved to $4\Delta - 4$ [**5**] and it is even conjectured that the correct value should be $\Delta + 2$ [**2, 6**] – therefore, obtaining a meaningful lower bound on $b(n)$ would require much more than just forbidding 2-colored 4-cycles.

Another pattern forbidden in visibility-blocking colorings is two three-colored triangles joined by two edges of the same color (i.e. vertices $v_1, \ldots, v_6$ such that $c(v_1v_2) = c(v_4v_5)$, $c(v_1v_3) = c(v_4v_5)$, $c(v_2v_3) = c(v_5v_6)$ and $c(v_1v_4) = c(v_3v_5)$). We will demonstrate that this pattern can be also avoided in an edge coloring of a complete graph using a linear number of colors. In fact, we will show this for a large class of patterns that includes all patterns forbidden in visibility-blocking colorings that we were able to think of. In order to formulate this result, we need to introduce some definitions.

A *pattern* is a pair (H, c), where H is a connected graph on at least 3 vertices and c is a coloring of the edges of H. Then, if c' is an edge-coloring of some graph G, then an *occurrence* of a pattern (H, c) in c' is an injective function $f \colon V(H) \to V(G)$ such that for every edge $uv \in E(H)$ we have $f(u)f(v) \in E(G)$ and for every two edges $uv, wx \in E(H)$, if $c(uv) = c(wx)$ then $c'(f(u)f(v)) = c'(f(w)f(x))$. We say that an edge-coloring c' *avoids* a pattern P if P does not occur in c'; also, c' *avoids* a set of patterns S if it avoid every patterns in S. For a pattern $P = (H, c)$, by $v(P)$ we denote the number of vertices in H, by $e(P)$ – the number of edges in H and by $c(P)$ – the number of colors that are used at least once by c. Our main result is the following.

Theorem 1. *Let S be a finite set of patterns, such that every pattern P in S satisfies $e(P) \geq v(P) + c(P) - 2$. Then, there exists a constant $C = C(S)$ such that for every n there is an edge-coloring of K_n that avoids S and uses at most Cn colors.*

2. Proof of the Main Theorem

We remark that it is relatively easy to obtain a weaker version of Theorem 1, where the condition of patterns in relaxed to $e(P) \geq v(P) + c(P)$ – it suffices to consider a random edge coloring of K_n with Cn colors and note that the expected number of pattern occurrences is less than one for an appropriate choice of C. This idea, however, does not yield a linear upper bound for forbidden patterns mentioned above and does not even work for proper edge colorings.

In fact, we will prove a stronger version of Theorem 1, that works for every graph and gives an upper bound linear in the maximum degree. In our proof we use the entropy compression method, that has originated from **[7]**, and is becoming a standard alternative to the Lovász Local Lemma. We give a sketch of the reasoning, leaving rigorous proofs of the two final claims to the reader.

Theorem 2. *Let S be a finite set of patterns, such that every pattern P in S satisfies $e(P) \geq v(P) + c(P) - 2$. Then, there exists a constant $C = C(S)$ such that for every graph G with maximum degree Δ there is an edge-coloring of G that avoids S and uses at most $C\Delta$ colors.*

Sketch of the proof. Take $s = |S|$ and suppose that $S = \{P_1, P_2, \ldots, P_s\}$. Let C be a constant such that $\frac{2e_{\max}}{C^{v(P_j)-2}} < \frac{1}{2s^{e_{\max}}}$ for every j, where $e_{\max}$ is the maximum number of edges in a pattern from S. Take any graph G of maximum degree Δ and let $M = M(G)$ be a sufficiently large integer.

The main idea of the proof is as follows: we suppose that no edge coloring of G that uses $C\Delta$ colors avoids S. Then, we use it to construct a procedure that uniquely encodes every sequence of length M over the set of Cn colors. Finally, we show that the number of possible encodings is less than $(Cn)^M$, which is a contradiction and shows that some edge coloring of G with Cn must avoid S.

The procedure takes as an input a sequence $c_1, c_2, \ldots, c_M$ over the set of $C\Delta$ colors and produces the following output.

- A sequence $D = d_1, d_2, \ldots$ of numbers from 0 to s
- A sequence $E = e_1, e_2, \ldots$ of numbers from 1 to $2e_{\max}$, where $e_{\max}$ is the maximum number of edges in a pattern from S
- A sequence $N = n_1, n_2, \ldots$ of numbers from 1 to Δ
- A sequence $X = x_1, x_2, \ldots$ over the set of $C\Delta$ colors
- A partial edge coloring K of G

In what follows, we will assume that all the orderings that will be used are fixed. Now we present the encoding procedure.

1. Initialize a partial edge coloring K of G where every edge is uncolored
2. For $i = 1, 2, \ldots, M$
   (a) If all edges of G are colored, report failure
   (b) Color the first uncolored edge $f = uv$ in G with c_i
   (c) If no pattern from S occurs, then append 0 to D and continue
   (d) Let P_j be a pattern from S that occurs in K (note that the occurrence contains f)

(e) Append to D the number j

(f) Append to E the number representing the edge of P that corresponds to f in the occurrence of P together with one of two orientations (uv or vu)

(g) Order the vertices of P such that the first vertex corresponds to u in the occurrence of P, the second corresponds to v, and every other has a neighbor that appears in the ordering earlier. Then, for every vertex w of P, except the first two in the ordering, pick some neighbor y of w in P that appears earlier in the ordering and append to N the number k, which has a meaning that the vertex of G corresponding to w in the occurrence of P is the k'th neighbor of the vertex corresponding to y.

(h) Append to X all colors that appeared in the occurrence of P, in an order determined by the fixed ordering of corresponding colors in P

(i) Uncolor all edges in the occurrence of P

3. Return (D, E, N, X, K)

The above procedure is designed so that resulting sequences D, E and N are sufficient to determine which edges of G have been colored and uncolored in each step (starting from the first), and then using X and K (and starting from the end) it is possible to determine colors used in each step.

Claim 1. *The sequence $c_1, c_2, \ldots, c_M$ is uniquely determined by the output (D, E, N, X, K) of the encoding procedure.*

Note that a pattern P can be found at step 2(d) once every $e(P)$ steps, so each such occurrence corresponds to $e(P)$ entries in D and clearly there are $s^{e(P)}$ possible values of $e(P)$ entries of D. In a similar manner, each occurrence of P gives one entry in E ($2e_{\max}$ possibilities), $v(P) - 2$ entries in N ($\Delta^{v(P)-2}$ possibilities) and $c(P)$ entries in X ($(C\Delta)^{c(P)}$ possibilities). On the other hand, each occurrence of P involves uncoloring $e(P)$ edges, which corresponds to $e(P)$ entries of the input sequence ($(C\Delta)^{e(P)}$ possibilities). However, note that $s^{e(P)} 2e\Delta^{v(P)-2} (C\Delta)^{c(P)}$ is smaller than $(C\Delta)^{e(P)}$ by the choice of C and an assumption $e(P) \geq v(P)+c(P)-2$. Similar reasoning for all occurrences of all patterns from S gives the following claim.

Claim 2. *The number of possible outputs (D, E, N, X, K) of the encoding procedure is less than $(C\Delta)^M$, provided that M is sufficiently large.*

Note that above claims give an immediate contradiction, which implies that that the encoding procedure must fail at step 2(a) for some input sequence. Hence, there exists an edge coloring of G with $C\Delta$ colors that avoids S and the proof is complete. □

3. Final remarks

Theorem 1 does not imply any bound on $b(n)$, but it suggests that $b(n)$ may be linear. It would imply a linear upper bound on $b(n)$ if we were able to characterize visibility-blocking colorings by a finite set of forbidden patterns satisfying the assumption $e(P) \geq v(P)+c(P)-2$. Although we doubt that such characterization exists, we failed to find any forbidden pattern P with $e(P) < v(P)+c(P)-2$, so it is an open problem.

We also think that avoiding patterns in edge colorings is interesting on its own. In particular, is there a nice characterization of patterns that can be avoided using linear number of colors? We suspect that Theorem 1 may be optimal in the following sense: if P is a pattern such that $e(P) < v(P)+c(P)-2$, then avoiding P requires superlinear number of colors (i.e. there is no constant C such that for all n there is an edge-coloring of K_n avoiding P that uses at most Cn colors).

Our method can be also used to show, that if $e(P) < v(P)+c(P)-2$, then P can be avoided using $O\left(n^{\frac{v(P)-2}{e(P)-c(P)}}\right)$ colors, but it is far from being tight. Proper edge colorings that avoid 2-edge colored path with 4 edges are known as star edge colorings, and they require not more than $n^{1+o(1)}$ colors – see Theorems 2.1 and 3.1 from [**4**].

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