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ON ORTHOGONAL SYMMETRIC CHAIN DECOMPOSITIONS

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Abstract. The *n-cube* is the poset obtained by ordering all subsets of $\{1,\dots,n\}$ by inclusion, and it can be partitioned into $\binom{n}{\lfloor n/2\rfloor}$ chains, which is the minimum possible number. Two such decompositions of the n-cube are called *orthogonal* if any two chains of the decompositions share at most a single element. Shearer and Kleitman conjectured in 1979 that the n-cube has $\lfloor n/2\rfloor+1$ pairwise orthogonal decompositions into the minimum number of chains, and they constructed two such decompositions. Spink recently improved this by showing that the n-cube has three pairwise orthogonal chain decompositions for $n\geq 24$. In this paper, we construct four pairwise orthogonal chain decompositions of the n-cube for $n\geq 60$. We also construct five pairwise *edge-disjoint* symmetric chain decompositions of the n-cube for $n\geq 90$, where edge-disjointness is a slightly weaker notion than orthogonality, improving on a recent result by Gregor, Jäger, Mütze, Sawada, and Wille.

1. Introduction

The *n-dimensional cube* Q_n, or n-cube for short, is the poset obtained by taking all subsets of $[n]:=\{1,\dots,n\}$, and ordering them by inclusion. This poset is sometimes also called the *subset lattice* or the *Boolean lattice*, and it is a fundamental and widely studied object in combinatorics. For illustration, Figure 1 shows the Hasse diagram of the 4-cube.

Clearly, Q_n is a graded poset with rank function given by the set sizes, and every maximal chain has size $n+1$. We refer to the family of all subsets of a fixed size $k\in\{0,\dots,n\}$ as the *kth level* of Q_n. It is easy to see that Q_n has a unique largest level $n/2$ for even n, and two largest levels $\lfloor n/2\rfloor$ and $\lceil n/2\rceil$ for odd n. We refer to these levels as *middle levels*. Sperner's classical theorem [**31**] asserts that each middle level is in fact a largest antichain of Q_n, i.e., Q_n has width $a_n:=\binom{n}{\lfloor n/2\rfloor}$. As a consequence, at least a_n many chains are needed to partition Q_n, and by Dilworth's theorem [**7**], a partition into this many chains indeed exists. De Bruijn, van Ebbenhorst Tengbergen, and Kruiswijk [**5**] first described an inductive construction of a partition of Q_n into a_n many chains that are all *symmetric* and *saturated*, i.e., every chain starts and ends in symmetric levels around

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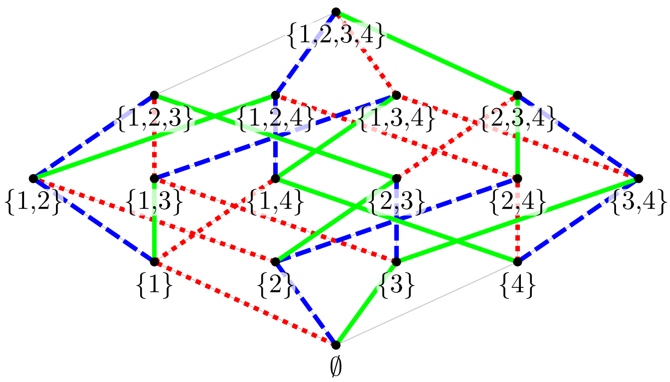

Figure 1. Hasse diagram of the 4-cube Q_4, with three pairwise orthogonal decompositions into 6 chains, highlighted by thick solid, dashed, and dotted lines.

the middle, and no chain skips any intermediate levels. Throughout this paper, we will refer to their decomposition as the *standard decomposition*. Lewin [**25**], Aigner [**2**], and White and Williamson [**34**] gave alternative descriptions of the standard decomposition via greedy matching algorithms as well as explicit local rules to follow the chains in the standard decomposition. The easiest-to-remember local rule using parenthesis matching was given by Greene and Kleitman [**13**]. The standard decomposition of Q_n was famously used by Kleitman [**21**] to prove the two-dimensional case of the Littlewood-Offord conjecture on signed sums of vectors [**26**] (later proved in all dimensions by Kleitman [**22**]).

Shearer and Kleitman [**30**] were the first to investigate chain decompositions of the n-cube that are different from the aforementioned standard decomposition. They proved that, when picking subsets $x, y \subseteq [n]$ at random, the probability that $x \subseteq y$ is at least $1/a_n$, for every probability distribution on Q_n. Their proof introduces the notion of orthogonal chain decompositions. Formally, two decompositions of Q_n into a_n (not necessarily symmetric or saturated) chains are called *orthogonal* if every two chains from the two decompositions have at most a single element of Q_n in common. For example, Figure 1 shows three pairwise orthogonal chain decompositions into 6 chains in Q_4. Shearer and Kleitman conjectured that Q_n admits $b_n := \lfloor n/2 \rfloor + 1$ pairwise orthogonal chain decompositions for all $n \geq 1$. As a warm-up exercise, we verified their conjecture for $n \leq 7$ with computer help. It is easy to check that there are at most b_n pairwise orthogonal decompositions (consider the node degrees in the Hasse diagram around the middle levels).

As a first step towards their conjecture, Shearer and Kleitman established the existence of *two* orthogonal chain decompositions for all $n \geq 2$. They proved this by showing that the standard decomposition and its *complement*, obtained by taking the complements of all sets with respect to the full set $[n]$, are almost-orthogonal. Formally, we say that two decompositions of Q_n into a_n symmetric and saturated chains are *almost-orthogonal* if every two chains from the two decompositions have at most a single element of Q_n in common, with the exception of the two unique chains of size $n+1$, which are only allowed to intersect in their minimal and maximal elements $\emptyset$ and $[n]$. It is straightforward to verify that for $n \geq 5$, every family of almost-orthogonal decompositions can be modified to orthogonal decompositions, by moving the empty set $\emptyset$ in all but one of the decompositions from the unique longest chain to a shortest chain, one decomposition at a time (see [**30, 32**] for details).

Recently, Spink [**32**] made the first progress towards the Shearer-Kleitman conjecture from 1979 by proving that Q_n has *three* pairwise orthogonal chain decompositions for $n \geq 24$. He actually showed that Q_n has three almost-orthogonal decompositions into symmetric and saturated chains, from which the result follows as described before.

Our results

Using Spink's product construction, we improve on his result as follows.

Theorem 1. *For all $n \geq 60$, the n-cube has four pairwise almost-orthogonal decompositions into symmetric and saturated chains, and consequently four pairwise orthogonal chain decompositions.*

A slightly weaker notion than almost-orthogonality was introduced in a recent paper by Gregor, Jäger, Mütze, Sawada, and Wille [**14**]. We refer to any cover relation $x \subseteq y$ as an *edge* (x, y) (y is one level above x), and we say that two decompositions of Q_n into a_n symmetric and saturated chains are *edge-disjoint* if the two decompositions do not share any edges. Equivalently, the two decompositions form edge-disjoint paths in the cover graph of Q_n, which is the graph formed by all cover relations. By this definition, every pair of almost-orthogonal chain decompositions is edge-disjoint, but not necessarily vice versa. The main application of edge-disjoint chain decompositions in [**14**] was to construct cycle factors in subgraphs of Q_n induced by an interval of levels around the middle, with the goal of generalizing the recent proof of the middle levels conjecture by Mütze [**27**] (see also [**15**]). It is also easy to check that Q_n admits at most b_n pairwise edge-disjoint chain decompositions. The authors of [**14**] conjectured that this bound can be achieved for all $n \geq 1$. They verified this conjecture for $n \leq 7$, and proved that Q_n has *four* pairwise edge-disjoint decompositions for $n \geq 12$. We improve on this result as follows.

Theorem 2. *For all $n \geq 90$, the n-cube has five pairwise edge-disjoint decompositions into symmetric and saturated chains.*

Unless stated otherwise, all chains we consider in the following are symmetric and saturated, and we will from now on omit those qualifications. Moreover, we refer to any decomposition of Q_n into symmetric and saturated chains as an *SCD*. Also, when referring to a family of pairwise almost-orthogonal or pairwise edge-disjoint SCDs, we will from now on omit the qualification 'pairwise'.

Table 1. Number of almost-orthogonal and edge-disjoint SCDs of Q_n for $n \leq 25$. Entries with * are new compared to the earlier results from [**32**] and [**14**]. For $n \leq 11$, the corresponding families of SCDs are provided electronically on the third authors' website [**1**] and on the arXiv [**6**]. For $n \geq 12$, they are obtained via the product constructions presented in [**32**] and [**14**].

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
almost-orthogonal SCDs	1	2	2	2	3	3*	4*	3*	3*	3	4*	3	3*	4*	3	3*	3	4*	3	3	4*	4*	3*	3	4*
edge-disjoint SCDs	1	2	2	3	3	4	4	4	4*	5*	6*	4	4	4	4	4	4	4	4	5*	5*	6*	4	4	4
upper bound $b_n = \lfloor n/2 \rfloor + 1$	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12	13	13

Small dimensions

Table 1 summarizes what is known for small values of n. Specifically, the table shows the maximum numbers of almost-orthogonal and edge-disjoint SCDs of Q_n that we know for $n \leq 25$, together with the upper bound b_n. As indicated in the table, we actually found six edge-disjoint SCDs of Q_{11}, which, using the product construction from [**14**], yields six edge-disjoint SCDs for all dimensions $n = 11k$, $k \in \mathbb{N}$. To extend this result to all but finitely many dimensions, thus improving Theorem 2, we would only need to find six edge-disjoint SCDs of Q_n for some dimension n not of this form. It is also interesting to note that there are *no* three almost-orthogonal SCDs of Q_4 (see [**32**]), i.e., in this case the trivial upper bound b_n cannot be achieved. Nevertheless, there are three orthogonal decompositions using non-symmetric chains in Q_4 – see Figure 1 – so this shows that not every family of orthogonal chain decompositions can be obtained from almost-orthogonal SCDs. Our lower bounds in the table for edge-disjoint SCDs differ from the upper bound b_n by 1 exactly for the dimensions $n = 8, 9, 10$ – see the values in the dotted box – and they cannot be improved by our methods.

Related work

There is a considerable amount of literature on partitioning the n-cube using possibly non-symmetric and/or non-saturated chains. One of the most interesting open problems in this direction is a well-known conjecture of Füredi [**12**] (cf. [**17**]), which asserts that Q_n can be decomposed into a_n (not necessarily symmetric or saturated) chains whose sizes differ by at most 1, so their size is $2^n/a_n$ rounded up or down, which is approximately $\sqrt{\pi n}(1 + o(1))$. Tomon [**33**] recently made some progress towards this conjecture, by showing that for large enough n, the n-cube can be decomposed into a_n chains whose size is between $0.8\sqrt{n}$ and $13\sqrt{n}$. Another remarkable result, recently shown by Gruslys, Leader, and Tomon [**18**], is that for large enough n, the n-cube can be partitioned into copies of any fixed poset P, provided that the number of elements of P is a power of 2 and that P has a unique minimal and maximal element. Pikurkho [**28**] showed that all edges of the n-cube can be partitioned into symmetric chains, but it is not clear whether some of those chains can be selected to form one or more SCDs.

Griggs, Killian, and Savage first constructed an explicit SCD of the necklace poset N_n [**16**] when the dimension n is a prime number, with the goal of constructing rotation-symmetric Venn diagrams for n curves in the plane. Their result was later generalized by Jordan [**20**] to all $n \in \mathbb{N}$, and to even more general quotients of Q_n by Duffus, McKibben-Sanders, and Thayer [**9**]. All these constructions for N_n proceed by taking suitable subchains from the standard SCD of Q_n.

As we employ SAT solvers in our work, we conclude this section by listing some recent results where they were used successfully to tackle difficult problems in (extremal) combinatorics, either by using them to find a solution, or to prove that no solution exists. Fujita [**11**] established a new lower bound $R(4, 8) \geq 58$ for the classical Ramsey numbers. Similarly, Dransfield, Liu, Marek, and Truszczyński [**8**] derived improved bounds for van der Waerden numbers (see also [**19**] and [**24**]). Another recent result that received considerable attention is described in the paper by Konev and Lisitsa [**23**] on the Erdős discrepancy conjecture. SAT solvers have also been used in the context of geometry, specifically for tackling Erdős-Szekeres type questions, see the papers by Balko and Valtr [**4**] and by Scheucher [**29**]. Moreover, with their help researchers were able to find new coil-in-the-box Gray codes [**36**] and to compute pairs of orthogonal diagonal Latin squares [**35**].

2. Proof ideas

We now outline the main ideas for proving Theorems 1 and 2. For details, see the preprint version of this paper [**6**].

Product constructions

We compute families of $s = 4$ almost-orthogonal and $s = 5$ edge-disjoint SCDs, for two cubes Q_a and Q_b of small coprime dimensions a and b. Specifically, these dimensions will be $(a, b) = (7, 11)$ and $(a, b) = (10, 11)$, respectively; see the shaded entries in Table 1. Using the product constructions presented in [**32**] and [**14**], we obtain s SCDs of the corresponding type in Q_n for all dimensions n that are non-negative integer combinations of a and b, in particular for all $n \geq (a-1)(b-1)$. This evaluates to $n \geq 60$ and $n \geq 90$ for the aforementioned pairs (a, b), respectively.

Problem reduction via the necklace poset

To find families of SCDs in cubes of small fixed dimension ($n = 7$, 10, and 11) that satisfy the desired constraints, we reduce the search space to a much smaller poset, the so-called necklace poset. Given a subset $x \subseteq [n]$, we write $\sigma(x)$ for the subset obtained from x by cyclically renaming elements $1 \to 2 \to \cdots \to n \to 1$. The family $\langle x \rangle$ of all subsets obtained by repeatedly applying σ to x is referred to as the *necklace* containing x. We say that the necklace $\langle x \rangle$ is *full* if $|\langle x \rangle| = n$, and *deficient* if $|\langle x \rangle| < n$. For example, for $n = 4$ the necklace $\langle \{1,3,4\} \rangle =$

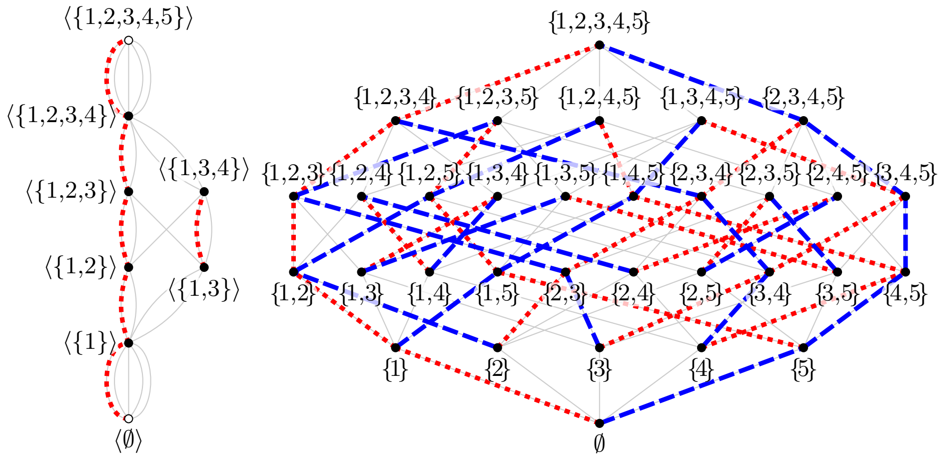

Figure 2. Unrolling an SCD of the necklace poset N_5 (left) to an SCD of the 5-cube (right). The SCD is dotted; full and deficient necklaces are indicated by filled or empty bullets, respectively. Complementing the resulting SCD of Q_5 yields another edge-disjoint SCD (dashed).

$\{\{1,3,4\},\{2,4,1\},\{3,1,2\},\{4,2,3\}\}$ is full, and the necklace $\langle\{1,3\}\rangle = \{\{1,3\},\{2,4\}\}$ is deficient. As the cardinality of any necklace divides n, $\langle\emptyset\rangle$ and $\langle[n]\rangle$ are the only deficient necklaces if n is a prime number. The *necklace poset* N_n is the set of all necklaces $\langle x\rangle$, $x \subseteq [n]$, and its cover relations are all pairs $(\langle x\rangle, \langle y\rangle)$ for which (x, y) is a cover relation in the n-cube; see Figure 2.

As σ preserves the set size, N_n inherits the level structure from Q_n, and notions such as symmetric chains and SCDs translate to N_n in the natural way. Moreover, as almost all necklaces of N_n are full, N_n is by a factor of $n(1 - o(1))$ smaller than Q_n, which turns out to be crucial for our computer searches for SCDs. We say that a chain in N_n is *unimodal* if its minimal and maximal element are necklaces of the same size (possibly deficient), and all other elements are full necklaces. In particular, if n is a prime number, then all chains are unimodal. We can unroll each unimodal chain in the necklace poset to multiple chains in Q_n as follows: Let $(y_0, \ldots, y_{k+1})$ be a unimodal chain in N_n with y_0 and y_{k+1} of size $d \leq n$. Then there are necklace representatives $(x_0, \ldots, x_{k+1})$ with $x_i \in y_i$ for $0 \leq i \leq k+1$, such that $\sigma^i(x_0, \ldots x_{k+1})$ for $i = 0, \ldots, d-1$, and $\sigma^i(x_1, \ldots, x_k)$ for $i = d, \ldots, n-1$, is a family of disjoint chains in Q_n that visit exactly all elements from $y_0 \cup \cdots \cup y_{k+1}$. Moreover, if we have an SCD of N_n consisting only of unimodal chains, then we can unroll each of its chains to obtain an SCD of Q_n; see Figure 2. We also introduce a suitable notion of edge multiplicities for the necklace poset (as indicated in Figure 2), which allows us to find multiple edge-disjoint SCDs in N_n simultaneously, and to unroll them to multiple edge-disjoint SCDs in Q_n.

Using SAT solvers

To search multiple edge-disjoint SCDs in the necklace poset N_n for some small fixed dimension n, we formulate this problem as a propositional formula in conjunctive normal form (CNF), and compute solutions using the SAT solvers Glucose [**3**] and MiniSat [**10**]. In our CNF formula, we use Boolean variables that indicate whether certain nodes and edges belong to a particular SCD, and we introduce clauses ensuring that in a satisfying variable assignment, all chains are unimodal and multiple SCDs are edge-disjoint. Once a valid variable assignment is found, we use incremental CNF augmentation to enforce the remaining properties, in particular almost-orthogonality of the unrolled SCDs in Q_n. Specifically, if we encounter a violation, we add an additional clause that prevents this particular configuration. We solve the augmented CNF using an incremental SAT solver, until we either find a feasible solution or obtain a formula with no satisfying assignment. This approach keeps the size of the generated CNFs and of the computation time small, as the solvers can reuse structural information of the CNFs, rather than recomputing a solution from scratch. The size of the formulas can be reduced further by prescribing some particularly nice SCDs.

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