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MOST PRINCIPAL PERMUTATION CLASSES, AND t-STACK SORTABLE PERMUTATIONS, HAVE NONRATIONAL GENERATING FUNCTIONS

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Abstract. We prove that for any fixed n, and for most permutation patterns q, the number $\mathrm{Av}_{n,\ell}(q)$ of q-avoiding permutations of length n that consist of ℓ skew blocks is a monotone decreasing function of ℓ. We then show that this implies that for most patterns q, the generating function $\sum_{n\geq 0} \mathrm{Av}_n(q)z^n$ of the sequence $\mathrm{Av}_n(q)$ of the numbers of q-avoiding permutations is not rational. Placing our results in a broader context, we show that for rational power series $F(z)$ and $G(z)$ with nonnegative real coefficients, the relation $F(z) = 1/(1 - G(z))$ is supercritical, while for most permutation patterns q, the corresponding relation is not supercritical.

1. Introduction

We say that a permutation p *contains* the pattern $q = q_1q_2 \dots q_k$ if there is a k-element set of indices $i_1 < i_2 < \cdots < i_k$ so that $p_{i_r} < p_{i_s}$ if and only if $q_r < q_s$. If p does not contain q, then we say that p *avoids* q. For example, $p = 3752416$ contains $q = 2413$, as the first, second, fourth, and seventh entries of p form the subsequence 3726, which is order-isomorphic to $q = 2413$. Let $\mathrm{Av}_n(q)$ be the number of permutations of length n that avoid the pattern q. In general, it is very difficult to compute, or even describe, the numbers $\mathrm{Av}_n(q)$, or their sequence as n goes to infinity. As far as the generating function $A_q(z) = \sum_{n\geq 0} \mathrm{Av}_n(q)z^n$ goes, there are known examples when it is algebraic, and known examples when it is not algebraic. The question whether $A_q(z)$ is always differentiably finite was raised in 1996 by John Noonan and Doron Zeilberger, and is still open.

In this abstract, we describe a proof for the theorem that for patterns $q = q_1q_2 \dots q_k$, where $k > 2$ and $\{q_1, q_k\} \neq \{1, k\}$, the generating function $A_q(z)$ is *never rational*. It is plausible to think that our result holds for the less than $1/[k(k-1)]$ of patterns of length k for which we cannot prove it. On the other hand, the statement obviously fails for the pattern $q = 12$, since for that q, we trivially have that $\mathrm{Av}_n(q) = 1$ for all n, so $A_q(z) = 1/(1 - z)$. The set of permutations of any length that avoid a given pattern q is often called a *principal permutation class*, explaining the title of this paper. As rational functions are differentiably

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finite, this paper excludes a small subset of differentiably finite power series from the set of possible generating functions of principal permutation classes.

Our main tool will be a theorem that is interesting on its own right. We say that a permutation p is *skew indecomposable* if it is not possible to cut p into two parts so that each entry before the cut is larger than each entry after the cut. For instance, $p = 3142$ is skew indecomposable, but $r = 346512$ is not as we can cut it into two parts by cutting between entries 5 and 1, to obtain $3465|12$.

If p is not skew indecomposable, then there is a unique way to cut p into *nonempty* skew indecomposable strings $s_1, s_2, \ldots, s_\ell$ of consecutive entries so that each entry of s_i is larger than each entry of s_j if $i < j$. We call these strings s_i the *skew blocks* of p. For instance, $p = 67|435|2|1$ has four skew blocks, while skew indecomposable permutations have one skew block.

The number of skew blocks of a permutation is of central importance for this paper. For permutations with no restriction, it is easy to prove that almost all permutations of length n are skew indecomposable. We prove that if q is a skew indecomposable pattern, and n is any fixed positive integer, then the number $\mathrm{Av}_{n,\ell}(q)$ of q-avoiding permutations of length n that consist of ℓ skew blocks is a monotone decreasing function of ℓ. That is, as the number ℓ of skew blocks increases, the number of q-avoiding permutations with ℓ skew blocks decreases.

Then we place our results into a broader context by discussing them from the perspective of supercritical relations, which we introduce in Definition 6.1. We show that our results imply that on the one hand, rational generating functions lead to supercritical relations (Theorem 6.3), while for most principal permutation classes, the corresponding relations defined by $A_q(z)$ are not supercritical (Theorem 6.2), proving that $A_q(z)$ is not rational.

Theorem 6.3 can be used to show that some other combinatorial generating functions are not rational. Present author has recently [**5**] used this technique to prove that for all t, the generating function counting t-stack sortable permutations of length n is not rational. Several equivalent definitions of such permutations can be found in [**4**]; here we will just give a quick one. Let $p = LnR$ be a permutation of length n, where L and R denote the possibly empty strings on the left and right of n. Then we define the map s recursively, by $s(p) = s(L)s(R)n$, with $s(1) = 1$. Finally, p is t-stack sortable if $s^t(p) = 12\ldots n$.

2. Preliminaries

Proposition 2.1. *Let q be any skew indecomposable pattern. If, for all positive integers n, the inequality*

$$\mathrm{Av}_{n,2}(q) \leq \mathrm{Av}_{n,1}(q) \tag{1}$$

holds, then for all positive integers n, and all positive integers ℓ, the inequality

$$\mathrm{Av}_{n,\ell+1}(q) \leq \mathrm{Av}_{n,\ell}(q)$$

holds.

This follows from the way in which the product of generating functions is computed.

If $q = q_1 q_2 \dots q_k$ is a pattern, let q^{rev} denote its reverse $q_k q_{k-1} \dots q_1$, and let q^c denote its complement $(k+1-q_1)(k+1-q_2)\dots(k+1-q_k)$. For instance, if $q = 25143$, then $q^{rev} = 34152$, and $q^c = 41523$. Recall that $\mathrm{Av}_n(q)$ denotes the number of permutations of length n that avoid q. It is then obvious that for all patterns q, the equalities $\mathrm{Av}_n(q) = \mathrm{Av}_n(q^{rev}) = \mathrm{Av}_n(q^c)$ hold. These equalities, and similar others, will be useful for us because of the following fact.

Proposition 2.2. *Let q and q' be two skew indecomposable patterns so that the equality*

$$\mathrm{Av}_n(q) = \mathrm{Av}_n(q') \tag{2}$$

holds for all $n \geq 1$. Then for all positive integers n, and for all positive integers $\ell \leq n$, the equality

$$\mathrm{Av}_{n,\ell}(q) = \mathrm{Av}_{n,\ell}(q') \tag{3}$$

holds.

3. The pattern 132

Next we mention the interesting fact that when $q = 132$, then in (1), equality holds if $n > 1$.

Lemma 3.1. *Let $n \geq 2$. Then the equality*

$$\mathrm{Av}_{n,2}(132) = \mathrm{Av}_{n,1}(132)$$

holds.

Indeed, just take a permutation counted by the left-hand side, and put its largest entry into the last position. It is easy to show that this map is injective.

Now Proposition 2.1 and Lemma 3.1, and the fact that $1 = \mathrm{Av}_{1,1}(132) > \mathrm{Av}_{1,2}(132) = 0$ together immediately imply the following.

Theorem 3.2. *For all positive integers n, and all positive integers $\ell \leq n-1$, the inequality*

$$\mathrm{Av}_{n,\ell+1}(132) \leq \mathrm{Av}_{n,\ell}(132)$$

holds.

4. The case containing most patterns

In the last section, we discussed a map that took a permutation with two skew blocks and moved its largest entry in its last position. For 132-avoiding permutations, this led to a bijection between two sets in which we were interested. In this section, we will replace 132 by a pattern q coming from a very large set of patterns. Furthermore, instead of moving the largest entry to the back, we will move the *last entry of the first skew block* to the end of the whole permutation.

(In the special case of $q = 132$, that entry happens to be the largest entry as well.) We will be able to show that this map is an *injection* from $\mathcal{A}v_{n,2}(q)$ to $\mathcal{A}v_{n,1}(q)$.

For the rest of this section, the pattern q is assumed to be skew indecomposable. Let us call a pattern $q = q_1q_2\dots q_k$ *good* if there does not exist a positive integer $i \leq k-1$ so that $\{q_{k-i}, q_{k-i+1}, \dots, q_{k-1}\} = \{1, 2, \dots, i\}$. That is, q is good if there is no proper segment immediately preceding its last entry whose entries would be the smallest entries of q. For instance, $q = 132$ and $q = 3142$ are good, but $q = 1324$ and $q = 35124$ are not, because of the choices of $i = 3$ in the former, and $i = 2$ in the latter. In particular, q is never good if $q_k = k$, because then we can choose $i = k - 1$.

Lemma 4.1. *Let q be a good pattern. Then for all positive integers n, the inequality*

$$\mathrm{Av}_{n,2}(q) \leq \mathrm{Av}_{n,1}(q)$$

holds.

In this case, we take the last entry of the first skew block, and place it in the last position, then prove that this map is an injection.

Now we are going to extend the reach of Lemma 4.1 to other patterns.

Lemma 4.2. *Let $q = q_1 \dots q_k$ be a skew indecomposable pattern so that $q_1 \neq 1$ or $q_k \neq k$ or both. The the inequality*

$$\mathrm{Av}_{n,2}(q) \leq \mathrm{Av}_{n,1}(q)$$

holds.

This can be done by symmetries such as reverse, complement, and inverse.

Lemma 4.2 does not cover patterns that start with their minimal element and end with their largest element, like 1324. However, if q is such a pattern, we can still prove the statement of Lemma 4.2 for q if q is Wilf-equivalent to a pattern q' that *is* covered by Lemma 4.2. Indeed, this is an immediate consequence of Proposition 2.2. So, for instance, the statement of Lemma 4.2 also holds for all monotone patterns $12\dots k$, since it is well-known [**1**] that $12\dots k$ is Wilf-equivalent to the pattern $12\dots(k-2)k(k-1)$.

The proof of the monotonicity result announced in the introduction is now immediate.

Theorem 4.3. *Let $q = q_1 \dots q_k$ be a skew indecomposable pattern so that at least one of the following conditions hold*

1. *$q_1 \neq 1$, or*
2. *$q_k \neq k$, or*
3. *$q_1 = 1$ and $q_k = k$, but q is Wilf-equivalent to a skew-indecomposable pattern in which the first entry is not 1 or the last entry is not k.*

Then the inequality

$$\mathrm{Av}_{n,\ell+1}(q) \leq \mathrm{Av}_{n,\ell}(q)$$

holds for all nonnegative integers n and all positive integers ℓ.

5. Why $A_q(z)$ is not rational

We can now prove the result mentioned in the title of the paper.

Theorem 5.1. *Let $q = q_1q_2 \dots q_k$ be a pattern so that either $\{1, k\} \neq \{q_1, q_k\}$, or q is Wilf-equivalent to a pattern $v = v_1v_2 \dots v_k$ so that $\{1, k\} \neq \{v_1, v_k\}$ Then the generating function $A_q(z)$ is not rational.*

Proof. First, note that we can assume that q is skew indecomposable. Indeed, if q is not, then q^{rev} is, and clearly, $A_q(z) = A_{q^{rev}}(z)$.

So let q be skew indecomposable, and let us assume that $A_q(z)$ is rational. Then the power series $A_{1,q}(z)$ is also rational. Let $r > 0$ be the radius of convergence of $A_{1,q}(z)$. We know that $r > 0$, since we know [**7**] that $\mathrm{A}v_{n,1}(q) \leq \mathrm{A}v_n(q) \leq c_q^n$ for some constant c_q. As the coefficients of $A_{1,q}(z)$ are all nonnegative real numbers, it follows from Pringsheim's theorem (Theorem IV.6 in [**6**]) that the positive real number r is a singularity of $A_{1,q(z)}$. As $A_{1,q}(z)$ is rational, r is a pole of $A_{1,q}(z)$, so $\lim_{z \to r} A_{1,q}(z) = \infty$. Therefore, there exists a positive real number $z_0 < r$ so that $A_{1,q}(z_0) > 1$. Therefore,

$$\sum_{n \geq 1} \mathrm{Av}_{n,1}(q) z_0^n = A_{1,q}(z_0) < A_{1,q}(z_0)^2 = A_{2,q}(z_0) = \sum_{n \geq 2} \mathrm{Av}_{n,2}(q) z_0^n,$$

contradicting the fact, proved in Theorem 4.3, that for each n, the coefficient of z^n in the leftmost powers series is at least as large as it is in the rightmost power series. ☐

The elegant argument in the previous paragraph is due to Robin Pemantle [**10**]. A significantly more complicated argument proves a stronger statement. The interested reader should consult [**2**] for details.

6. Broader context: Supercritical relations

We will place our results into the broader context of *supercritical relations*. Readers who are interested to learn more about this subject are invited to consult sections V.2 and VI.9 of [**6**].

Definition 6.1. Let F and G be two generating functions with nonnegative real coefficients that are analytic at 0, and let us assume that $G(0) = 0$. Then the relation

$$F(z) = \frac{1}{1 - G(z)}$$

is called *supercritical* if $G(R_G) > 1$, where R_G is the radius of convergence of G.

Note that as the coefficients of $G(z)$ are nonnegative, $G(R_G) > 1$ implies that $G(\alpha) = 1$ for some $\alpha \in (0, R_G)$. So, if the relation between F and G described above is supercritical, then the radius of convergence of F is less than that of G, therefore, the exponential growth rate of the coefficients of F is larger than that of G.

Theorem 6.2. *Let q be any permutation pattern satisfying the conditions of* Theorem 4.3. *Then the relation*

$$A_q(z) = \frac{1}{1 - A_{1,q}(z)}$$

is not supercritical.

Proof. It is immediate from Theorem 4.3 that we have

$$\mathrm{Av}_n(q) = \sum_{\ell=1}^{n} \mathrm{Av}_{n,\ell}(q) = n\mathrm{Av}_{n,1}(q),$$

implying that the sequences $\mathrm{Av}_n(q)$ and $\mathrm{Av}_{n,1}(q)$ have the same exponential order. By Corollary 6.2, that means that the relation between their generating functions cannot be supercritical. □

On the other hand, combinatorial generating functions that are rational lead to supercritical relations, as the following extension of Theorem 5.1 shows.

Theorem 6.3. *Let $G(z)$ be a rational power series with nonnegative real coefficients that satisfies $G(0) = 0$. Then the relation*

$$F(z) = \frac{1}{1 - G(z)}$$

is supercritical.

Proof. If $G(z)$ is a polynomial, then $R_G = \infty$, so $G(R_G) = \infty > 1$, and our claim is proved. Otherwise, $G(z)$ is a rational function that has at least one singularity, and all its singularities are poles. Let R_G be a singularity of smallest modulus. Then $G(R_G) = \infty > 1$, completing our proof. □

Now we see that Theorem 5.1 immediately follows from the two results in this section. Indeed, if q is a pattern satisfying the conditions of Theorem 4.3, then $A_q(z)$ cannot be rational, because if it was, then so would be $A_{1,q}(z)$. Therefore, by Theorem 6.3, the relation $A_q(z) = \frac{1}{1-A_{1,q}(z)}$ would be supercritial, but we know by Theorem 6.2 that it is not.

7. Further directions

It goes without saying that it is an intriguing problem to prove Lemma 4.2 for the remaining patterns. Of course, Theorem 5.1 could possibly be proved by other means, but numerical evidence seems to suggest that Theorem 5.1 will hold even for patterns that start with their minimum entry and end in their largest entry. Interestingly, the shortest patterns for which we cannot prove Theorem 5.1 are 1324 and 4231, which also happen to be the shortest patterns for which no exact formula is known for $\mathrm{Av}_n(q)$.

It is important to point out that our results do not hold at all for permutation classes that are generated by more than one pattern. For instance, let $\mathrm{Av}_n(123, 132)$ denote the number of permutations of length n that avoid both 123 and 132. It is then easy to prove that $\mathrm{Av}_n(123, 132) = 2^{n-1}$, so $A_{123,132}(z) = (1-z)/(1-2z)$, a rational function. Note that in this case, $\mathrm{Av}_{n,1}(123, 132) = 1$, since the only such permutation is $(n-1)(n-2)\dots 1n$, while $\mathrm{Av}_{n,2}(123, 132) = n-1$, so Lemma 4.2 does not hold.

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