

AUTOMORPHISM GROUPS OF EDGE-TRANSITIVE MAPS

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ABSTRACT. For each of the 14 classes of edge-transitive maps described by Graver and Watkins, necessary and sufficient conditions are given for a group to be the automorphism group of a map, or of an orientable map without boundary, in that class. Extending earlier results of Širáň, Tucker and Watkins, these are used to determine which symmetric groups S_n can arise in this way for each class. Similar results are obtained for all finite simple groups, building on work of Leemans and Liebeck, Nuzhin and others on generating sets for such groups. It is also shown that each edge-transitive class realises finite groups of every sufficiently large nilpotence class or derived length, and also realises uncountably many non-isomorphic infinite groups.

1. INTRODUCTION

Maps are embeddings of graphs in surfaces, with simply-connected faces; those with a high degree of symmetry are of particular interest. In 1997 Graver and Watkins [3] partitioned edge-transitive maps \mathcal{M} into 14 classes, distinguished by the isomorphism class of the quotient map $\mathcal{M}/\text{Aut } \mathcal{M}$; in that year, Wilson [22] gave a similar classification.

These classes T correspond bijectively to the 14 isomorphism classes of maps $\mathcal{N}(T)$ with one edge. These are shown in Figure 1, where all maps are on the closed disc, except 2^Pex , 5 and 5^* on the sphere, 4^P on the Möbius band, and 5^P on the real projective plane. They include the class 1 of regular maps (for which $\text{Aut } \mathcal{M}$ is transitive on flags), the class 2^Pex of chiral maps (the non-regular orientable maps for which the orientation-preserving subgroup $\text{Aut}^+ \mathcal{M}$ is transitive on arcs), and the class 3 of just-edge-transitive maps (for which $\text{Aut } \mathcal{M}$ is transitive on edges but not vertices or faces). The duals of the maps in class 2 form class 2^* , while the Petrie duals of the latter form class 2^P ; the same applies to classes 2ex , 4 and 5, while classes 1 and 3 are invariant under these two operations.

This note describes the automorphism groups of maps in these 14 classes, continuing work begun by Širáň, Tucker and Watkins in [18]. Because of space constraints, results can only be stated here. The method used to prove them is briefly outlined in the final section; for full details, see [7].

Received May 27, 2019.

2010 *Mathematics Subject Classification.* Primary 05C10; Secondary 20B25.

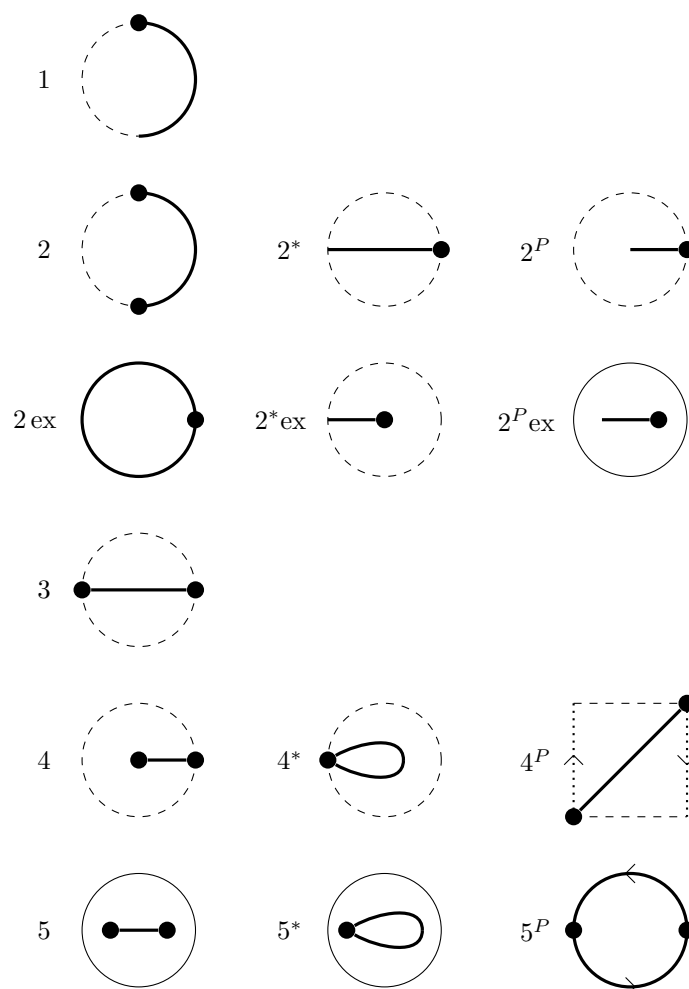


Figure 1. The basic maps $\mathcal{N}(T)$ for the 14 edge-transitive classes T .

2. FINITE EDGE-TRANSITIVE MAPS AND GROUPS

Širáň, Tucker and Watkins [18] showed that for each integer $n \geq 11$ with $n \equiv 3$ or $11 \pmod{12}$, there are finite, orientable, edge-transitive maps \mathcal{M} in each of the 14 edge-transitive classes T , with $\text{Aut } \mathcal{M}$ isomorphic to the symmetric group S_n . This is generalised as follows, where $\mathcal{G}(T)$ denotes the set of groups $G \cong \text{Aut } \mathcal{M}$ for some map \mathcal{M} in class T :

Theorem 2.1. *A symmetric group S_n , an alternating group A_n , or a projective special linear group $L_2(q) = \text{PSL}_2(q)$ is in $\mathcal{G}(T)$ if and only if it satisfies the corresponding condition in Table 1.*

Table 1. Groups S_n , A_n and $L_2(q)$ in sets $\mathcal{G}(T)$.

Class T	S_n	A_n	$L_2(q)$
1	$n \geq 1$	$n = 1, 2, 5$ or $n \geq 9$	$q \neq 3, 7, 9$
2, 2^* , 2^P	$n \geq 2$	$n \geq 5$	$q \neq 3$
2 ex, 2^* ex, 2^P ex	$n \geq 6$	$n \geq 8$	no q
3	$n \geq 2$	$n \geq 5$	$q \neq 3$
4, 4^* , 4^P	$n \geq 2$	$n \geq 4$	every q
5, 5^* , 5^P	$n \geq 6$	$n \geq 7$	no q

In the case $T = 1$, Sjerve and Cherkassoff [19] dealt with these groups together with $PGL_2(q)$, while Nuzhin [12, 13, 14, 15] dealt with A_n and the simple groups of Lie type, including $L_2(q)$. In most cases, all but finitely many of these groups are realised in each class T , the exceptions being six classes where no groups $L_2(q)$ arise. The proof, here and for other results stated below, follows [18] in using necessary and sufficient conditions for a group to be in the various sets $\mathcal{G}(T)$: for instance, when $T = 1$ these require the group to have generators r_i ($i = 0, 1, 2$) satisfying $r_i^2 = (r_0 r_2)^2 = 1$. (See §5 for the other classes.) It is then a routine (though often lengthy) matter to apply these conditions to the groups in Theorem 2.1.

The groups A_n ($n \geq 5$) and $L_2(q)$ ($q \geq 4$) are all simple. More generally, it is of interest to determine, for each class T , which non-abelian finite simple groups are in $\mathcal{G}(T)$. Here we use ATLAS notation [2] for simple groups:

Theorem 2.2. *A non-abelian finite simple group is in $\mathcal{G}(T)$ if and only if it is not one of the exceptions listed in the corresponding row of Table 2.*

Table 2. Non-abelian finite simple groups not in sets $\mathcal{G}(T)$.

Class T	Non-abelian finite simple groups $G \notin \mathcal{G}(T)$
1	$L_3(q), U_3(q), L_4(2^e), U_4(2^e), U_4(3), U_5(2), A_6, A_7, M_{11}, M_{22}, M_{23}, McL$
2, 2^* , 2^P	$U_3(3)$
2 ex, 2^* ex, 2^P ex	$L_2(q), L_3(q), U_3(q), A_7$
3	—
4, 4^* , 4^P	—
5, 5^* , 5^P	$L_2(q)$

Corollary 2.3. *Every non-abelian finite simple group is isomorphic to the automorphism group of an edge-transitive map.*

The entry for $T = 1$ is due to Nuzhin [12, 13, 14, 15] and others, through their answer to Mazurov's Kurovka Notebook question [1, Problem 7.30] asking which

non-abelian finite simple groups can be generated by three involutions, two of them commuting; the unitary groups $U_4(3)$ and $U_5(2)$ were recently added to the published lists after a computer search by Mačaj [10] revealed their omission. The solution for $T = 2^P \text{ex}$ has recently been determined by Leemans and Liebeck [9] in the equivalent context of abstract polyhedra, and a simple duality argument extends their result to 2ex and 2^*ex . The entries for the remaining classes are apparently new.

The exceptions for these ten classes are easily explained. The unitary group $U_3(3)$ is not in $\mathcal{G}(T)$ for the classes $T = 2$, 2^* and 2^P since groups realised in such classes must be generated by at most three involutions, and Wagner [20] has shown that this group requires four. A result of Singerman [17] shows that, for each generating pair for $L_2(q)$, there is an automorphism inverting both generators; such an automorphism is forbidden for the classes $T = 5$, 5^* and 5^P , so $L_2(q) \notin \mathcal{G}(T)$. The exceptions for $T = 2 \text{ex}$ and 2^*ex are the same as those found in [9] for $T = 2^P \text{ex}$.

Simple arguments show that if a group is realised in class 1 or 2^Pex it is also realised in various other classes, so to show that a group is in $\mathcal{G}(T)$ one can concentrate on those groups not in $\mathcal{G}(1)$ or $\mathcal{G}(2^P \text{ex})$ (for finite simple groups, these are the exceptions in the first and third rows of Table 2). To realise such groups, more direct arguments are required, finding specific generators and then showing that these do not admit forbidden automorphisms.

Let $\mathcal{G}^+(T)$ be the set of groups $G \cong \text{Aut } \mathcal{M}$ for an orientable map without boundary in class T . All maps in class $T = 2^P \text{ex}$, 5 or 5^* have these properties, so $\mathcal{G}^+(T) = \mathcal{G}(T)$. For the other classes, each group in $\mathcal{G}^+(T)$ must have a subgroup of index 2, so in particular no simple group (other than C_2 for $T = 1$) is in $\mathcal{G}^+(T)$. For example, by Theorem 2.1 no group $L_2(q)$ is in $\mathcal{G}^+(T)$ for any T . For S_n and A_n we have:

Theorem 2.4. *A group S_n or A_n is in $\mathcal{G}^+(T)$ if and only if it satisfies the corresponding condition in Table 3.*

Table 3. Groups S_n and A_n in sets $\mathcal{G}^+(T)$.

Class T	$S_n \in \mathcal{G}^+(T)$	$A_n \in \mathcal{G}^+(T)$
1	$n \neq 1, 5, 6$	no n
2, 2^*	$n \neq 1, 2, 5, 6$	no n
2^P	$n \geq 3$	no n
2ex , 2^*ex	$n \geq 7$	no n
2^Pex	$n \geq 6$	$n \geq 8$
3	$n \geq 3$	no n
4, 4^* , 4^P	$n \geq 3$	no n
5, 5^*	$n \geq 6$	$n \geq 7$
5^P	$n \geq 6$	no n

Theorem 2.5. $\mathcal{G}(T)$ contains a finite group of nilpotence class c or derived length l if and only if c or l satisfy the corresponding condition in Table 4.

Table 4. Nilpotence class and derived length.

Class T	Nilpotence class c	Derived length l
$2\text{ex}, 2^*\text{ex}, 2^P\text{ex}$	$c \geq 5$	$l \geq 2$
$5, 5^*, 5^P$	$c \geq 2$	$l \geq 2$
All other T	$c \geq 1$	$l \geq 1$

There has been recent interest [6] in arc-transitive maps. These results can be applied to them by restricting attention to the arc-transitive classes $T = 1, 2^*, 2^P, 2^*\text{ex}$ and 2^Pex . For instance, every non-abelian finite simple group except $U_3(3)$ is the automorphism group of an arc-transitive map.

3. INFINITE EDGE-TRANSITIVE MAPS AND GROUPS

The proof of a classic result of Neumann [11] is adapted to prove:

Theorem 3.1. *Each of the 14 edge-transitive classes T contains 2^{\aleph_0} maps \mathcal{M} with empty boundary and with mutually non-isomorphic automorphism groups $\text{Aut } \mathcal{M}$.*

Similarly, an embedding theorem of Schupp [16] is used to prove:

Theorem 3.2. *For each of the 14 edge-transitive classes T , every countable group C is isomorphic to a subgroup of $\text{Aut } \mathcal{M}$ for some map \mathcal{M} in T .*

Adapting Grigorchuk's construction [4, 5] of groups with intermediate growth (faster than any polynomial, slower than any exponential function), we have:

Theorem 3.3. *Each of the 14 edge-transitive classes T contains 2^{\aleph_0} maps \mathcal{M} such that \mathcal{M} and $\text{Aut } \mathcal{M}$ have intermediate growth.*

4. MAPS WITH NON-EMPTY BOUNDARY

Let us define a class T to be *void* or *tame* if it contains no maps with non-empty boundary, or if it contains such maps and they all have dihedral automorphism groups; otherwise it is *wild*.

Theorem 4.1. *Of the 14 classes of edge-transitive maps,*

- *six, namely $2\text{ex}, 2^*\text{ex}, 2^P\text{ex}, 5, 5^*$ and 5^P , are void,*
- *four, namely $1, 2, 2^*$ and 2^P , are tame, and*
- *four, namely $3, 4, 4^*$ and 4^P , are wild.*

For each tame class, one can classify its maps with non-empty boundary (see [7]); most are on the closed disc, with a few on the annulus, Möbius band or doubly infinite strip. For the wild classes, while it is possible to give general descriptions, such a classification is about as difficult as that of classifying all regular maps, and thus impossible with current techniques.

5. OUTLINE OF THE METHOD

As shown in [7, 8], maps \mathcal{M} can be identified with permutation representations $\Gamma \rightarrow \text{Sym } \Phi$ of the group

$$\Gamma = \langle R_i \ (i = 0, 1, 2) \mid R_i^2 = (R_0 R_2)^2 = 1 \rangle,$$

where each R_i acts on the set Φ of flags (incident vertex-edge-face triples) of \mathcal{M} by changing the i -dimensional component of each flag (whenever possible) while preserving the other two. Fixed points of R_i correspond to flags on the boundary. Now \mathcal{M} is connected (as we always assume) if and only if Γ is transitive on Φ , in which case \mathcal{M} corresponds to a conjugacy class of *map subgroups* M of Γ , the stabilisers of flags. Then $\text{Aut } \mathcal{M} \cong N_\Gamma(M)/M$ where $N_\Gamma(M)$ is the normaliser of M in Γ , and \mathcal{M} is edge-transitive if and only if $\Gamma = N_\Gamma(M)E$ where E is the Klein four-group $\langle R_0, R_2 \rangle \leq \Gamma$. The 14 edge-transitive classes T thus correspond to the 14 conjugacy classes of subgroups $N = N(T) \leq \Gamma$ such that $\Gamma = NE$, so that \mathcal{M} is in T if and only if $N_\Gamma(M) = N(T)$; then \mathcal{M} is a regular covering, by $\text{Aut } \mathcal{M}$, of the basic map $\mathcal{N}(T)$ corresponding to the map subgroup $N(T)$ of Γ (see Figure 1). For instance $N(1) = \Gamma$, and $\mathcal{N}(1)$ is the trivial map with one flag.

Table 5. Groups $N(T)$ and forbidden automorphisms.

Class T	$N(T)$	forbidden automorphisms
1	$(C_2 \times C_2) * C_2$	none
2	$C_2 * C_2 * C_2$	s_1 and s_2 transposed, s_3 fixed
2 ex	$C_2 * C_\infty$	s_1 fixed, s_2 inverted
3	$C_2 * C_2 * C_2 * C_2$	double transpositions of generators s_i
4	$C_2 * C_2 * C_\infty$	s_1 and s_2 transposed, s_3 inverted
5	$C_\infty * C_\infty$	s_1 and s_2 inverted, transposed or both

The subgroups $N(T)$, having finite index in the finitely presented group Γ , all have finite presentations, which are easily found. The automorphism groups $A = \text{Aut } \mathcal{M}$ for maps \mathcal{M} in T are all obtained as quotients $N(T)/M$ of $N(T)$ by normal subgroups M of $N(T)$. However, there is an additional constraint: this construction gives $N_\Gamma(M) \geq N(T)$, but we require equality here, which is equivalent (when $T \neq 1$) to A not admitting certain ‘forbidden automorphisms’ induced by larger subgroups of Γ than $N(T)$ which might normalise M . A group A is thus contained in $\mathcal{G}(T)$ if and only if it has generators satisfying relations corresponding to the defining relations of $N(T)$, so that it is a quotient of $N(T)$, and also admits none of the forbidden automorphisms associated with the class T . In this way one can determine which groups A are in $\mathcal{G}(T)$ for each class T . This task is eased by the fact that the 14 conjugacy classes of subgroups $N(T)$ form just six orbits under the action of the outer automorphism group $\text{Out } \Gamma \cong S_3$ (corresponding to Wilson’s group S_3 of map operations generated by duality and Petrie duality [8, 21], giving the six rows in Figure 1), so that it is sufficient to consider just one representative from each orbit. These are shown in Table 5, where the groups $N(T)$ are decomposed as free products, and the

forbidden automorphisms for $T \neq 1$ are specified by their effect on the images $s_i \in A$ of generators of successive cyclic free factors of $N(T)$. See [7] for full details.

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