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ON THE CHROMATIC INDEX OF COMPLEMENTARY PRISMS

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Abstract. This paper addresses the edge-colouring problem restricted to the graph class of complementary prisms. This graph class includes the Petersen graph, a very important and widely studied graph in the context of graph edge-colouring and remarkable related open questions, such as the Overfull Conjecture. We prove that all non-regular complementary prisms are *Class 1* and we conjecture that the only *Class 2* regular complementary prism is the Petersen graph. We present evidences for this conjecture.

1. Introduction

This work approaches the edge-colouring problem in the class of complementary prisms. This noteworthy graph class, which includes some important graphs such as the Petersen graph, was introduced in [**3**] and has been studied specially in the context of domination problems. Being G any graph on a non-empty set of vertices, the complementary prism $G\overline{G}$ is the graph obtained from the graphs G and its complement $\overline{G}$ by connecting with an edge each vertex in G to its corresponding vertex in $\overline{G}$ (see Fig. 1).

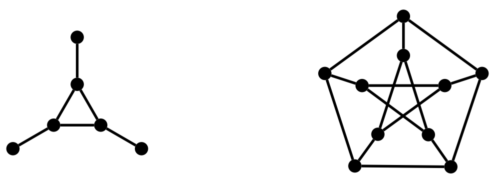

Figure 1. The complementary prisms $K_3\overline{K_3}$ and $C_5\overline{C_5}$ (the *Petersen graph*)

All graphs considered in this work are simple (i.e. undirected, loopless, without multiple edges). The set of neighbours of a vertex u in a graph G and the set of edges incident to u in G are denoted $N_G(u)$ and $\partial_G(u)$, respectively. For any

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$X \subseteq V(G)$, we define the set $\partial_G(X) := \{uv \in E(G) : u \in X \text{ and } v \notin X\}$. If $\emptyset \neq X \neq V(G)$, we say that $\partial_G(X)$ is the *cut* induced by X in G. If G is a regular graph, we use $d(G)$ to denote the degree of any vertex.

When particularly dealing with a complementary prism $G\overline{G}$, we convention that n is the order of G and $N = 2n$ is the order of $G\overline{G}$. Also, the parameters written without parenthesis, such as Δ or d, are always assumed to refer to the graph G, not to be confused with the parameters of $G\overline{G}$. We also assume without loss of generality that $\Delta(G) \geq \Delta(\overline{G})$, which implies $\Delta(G\overline{G}) = \Delta(G) + 1$.

A *k-edge-colouring* of a graph G is a function $\varphi\colon E(G) \to \mathscr{C}$ such that $\mathscr{C}$ is a set with k colours and $\varphi(e) \neq \varphi(f)$ for every pair (e, f) of distinct adjacent edges. By Vizing's Theorem **[9]**, the *chromatic index* of G (i.e. the least k for which G is *k-edge-colourable*) is either the maximum degree $\Delta(G)$ of G or it is $\Delta(G)+1$, and G is said to be *Class 1* in the former case or *Class 2* in the latter.

In an edge-colouring we say that some colour α is *missing* at some vertex u if no edge incident to u is assigned α. If we have an edge-colouring $\varphi\colon E(G) \setminus \{uv\} \to \mathscr{C}$ of $G - uv$ for some $uv \in E(G)$, and if u and all the neighbours of u in G miss at least one colour of $\mathscr{C}$ each, then by *Vizing's Recolouring Procedure* **[9]** we can obtain an edge-colouring of the graph G using only the colours of $\mathscr{C}$.

A *critical* graph is a connected *Class 2* graph G such that $\chi'(G-e) < \chi'(G)$ for all $e \in E(G)$. If G is a *Class 2* graph, then for every $k \in \{2, \ldots, \Delta(G)\}$ the graph G has a critical subgraph H with $\Delta(H) = k$ **[10]**. *Vizing's Adjacency Lemma*, also proved in **[10]**, states that for every edge uv of a critical graph G, the number of vertices of degree $\Delta(G)$ adjacent to u in G is at least $\max\{2, \Delta(G) - d_G(v) + 1\}$.

An n-vertex graph with maximum degree Δ is said to be *overfull* if it has more than $\Delta\lfloor n/2 \rfloor$ edges. A graph G is said to be *subgraph-overfull* (shortly, *SO*) if it has an *overfull* subgraph H with $\Delta(H) = \Delta(G)$. Clearly, every *SO* graph G is *Class 2*. There has been much work (e.g. **[1, 4, 8]**) in the last 30 years aimed at identifying graph classes wherein being *Class 2* is equivalent to being *SO*, since subgraph-overfullness is a polynomial-time testable property **[7]**. Moreover, the *Overfull Conjecture* states that this equivalence holds for all n-vertex graphs with maximum degree $\Delta > n/3$ **[1, 2]**.

It is important to investigate graph classes wherein the chromatic index is not related to the *SO* property, but to something else, since such investigation can help to improve the state of the art on the combinatorial structure of the edge-colouring problem. Graphs which are *Class 2* but not *SO* seem to be rare and the Petersen graph is one of them. Actually, we prove in Sect. 3 that no complementary prism can be *SO*, which justifies the importance of studying the edge-colouring problem in this graph class.

We prove in Sect. 2 that all non-regular complementary prisms are *Class 1*. Concerning regular complementary prisms, the problem seems to be more challenging. It is clear that $G\overline{G}$ is a regular complementary prism if and only if G and $\overline{G}$ are d-regular graphs with odd n and even $d = (n-1)/2$, which implies that G and $\overline{G}$ are both overfull and that $N \equiv 2 \pmod 8$. Therefore, the smallest regular complementary prism is the K_2 and the second smallest is the Petersen graph. The next regular complementary prisms have order 18, the next ones have order 26, and so on. In particular, because the number of 4-regular graphs on 9 vertices is 16, being 4 of them self-complementary, there are 10 regular complementary prisms on 18 vertices, which we have verified to be all *Class 1* using a backtracking algorithm (see Sect. 4 for the edge-colourings found). Since we have verified the same for 10 000 randomly generated regular complementary prisms on 26 vertices[1] (sampled with replacement), and since we prove in Sect. 3 that no complementary prism can be *SO*, we propose the following conjecture.

Conjecture 1. The Petersen graph is the only *Class 2* complementary prism.

2. All non-regular complementary prisms are *Class 1*

In this section we prove that every non-regular complementary prism $G\overline{G}$ admits a $(\Delta(G)+1)$-edge-colouring (recall that we have assumed $\Delta(G) \geq \Delta(\overline{G})$ without loss of generality). Since $G\overline{G}$ is non-regular if and only if (i) $\Delta(G) \neq \Delta(\overline{G})$ or (ii) G is non-regular, we split the proof in the two lemmas below, each handling one of these cases.

Lemma 1. *If G is a graph such that $\Delta(G) \neq \Delta(\overline{G})$, then $G\overline{G}$ is Class 1.*

Proof. Since we have assumed $\Delta(G) \geq \Delta(\overline{G})$, we can use Vizing's Theorem to start colouring the edges in G and in $\overline{G}$ using a colour set $\mathscr{C}$ with cardinality $\Delta(G)+1 = \Delta(G\overline{G})$.

Let M be the perfect matching formed by the edges connecting the vertices of G and their corresponding vertices in $\overline{G}$. We shall colour the edges of M one at a time. At each step, let $uv \in M$ be the edge under consideration and let H be the subgraph of $G\overline{G}$ induced by the edges which have been coloured up to this step. Assuming without loss of generality that $u \in V(G)$, all neighbours of v in H are in $\overline{G}$ and thus have degree less than $\Delta(G)+1$ in H, which implies that all vertices in the set $\{u, v\} \cup N_H(v)$ miss at least one colour of $\mathscr{C}$ each. The proof is then concluded by Vizing's Recolouring Procedure. $\square$

Lemma 2. *If G is a non-regular graph such that $\Delta(G) = \Delta(\overline{G})$, then $G\overline{G}$ is Class 1.*

Proof. We assume $n \geq 4$, since the K_1 is the only graph on at most 3 vertices whose maximum degree equals the maximum degree of its complement. We know that the vertices of minimum degree in G correspond to the vertices of maximum degree in $\overline{G}$ and vice versa, which implies that G and $\overline{G}$ have the same minimum degree δ, as they have the same maximum degree Δ. Hence,

$$\Delta(G) + \delta(G) + 1 = n\,, \tag{1}$$

and, since $\Delta \geq \delta + 1$ because G is not regular,

$$\Delta \geq \frac{n}{2} \qquad \text{and} \qquad \delta \leq \frac{n-2}{2}\,. \tag{2}$$

[1]There are 367 860 6-regular graphs on 13 vertices **[5]**, so there must be at least 183 930 regular complementary prisms on 26 vertices.

Let us assume for the sake of contradiction that $G\overline{G}$ is *Class 2*. So, let H be a critical subgraph of $G\overline{G}$ with the same maximum degree as $G\overline{G}$. Since $\Delta(H) = \Delta+1$, we know that H must contain at least one edge of M incident to a vertex of degree Δ in G or in $\overline{G}$, being M the perfect matching between G and $\overline{G}$ as in the proof of Lemma 1. Moreover, the vertices of degree $\Delta+1$ in H cannot be all in G or all in $\overline{G}$, otherwise H would be *Class 1* by taking a $(\Delta+1)$-edge-colouring of $H - (M \cap E(H))$ and then applying Vizing's Recolouring Procedure at each edge of $M \cap E(H)$ in order to colour it, as in the proof of Lemma 1.

So, H must contain two edges uu' and vv' of M such that the vertices u and v' have degree Δ in G and $\overline{G}$, respectively. Likewise, the vertices u' and v have degree δ in G and $\overline{G}$, respectively. Therefore, by Vizing's Adjacency Lemma, the vertex u must be adjacent in G to at least $\Delta(H) - d_H(u') + 1 \geq \Delta - \delta + 1$ vertices of degree Δ in G, as the vertex v' must also be adjacent in $\overline{G}$ to at least $\Delta(H) - d_H(v) + 1 \geq \Delta - \delta + 1$ vertices of degree Δ in $\overline{G}$. As u and v' are themselves vertices of degree Δ in $G \cup \overline{G}$, the total number of vertices of degree Δ in $G \cup \overline{G}$ must be at least $2(\Delta - \delta + 2)$, having all other vertices of $G \cup \overline{G}$ degree at least δ. Ergo, in view of (1),

$$\begin{aligned}
|E(G\overline{G}) - M| &= \frac{1}{2} \sum_{x \in V(G)} d_G(x) + \frac{1}{2} \sum_{y \in V(\overline{G})} d_{\overline{G}}(y) \\
&\geq (\Delta - \delta + 2)\Delta + (n - \Delta + \delta - 2)\delta \\
&= (\Delta - \delta + 2)\Delta + (2\delta - 1)\delta \\
&= (\Delta + \delta)^2 - (\Delta - \delta + 1)\delta \\
&= (n-1)^2 - (\Delta - \delta + 1)\delta\,.
\end{aligned}$$

We know that $|E(G\overline{G}) - M| = n(n-1)/2$, since the identification of the corresponding vertices in G and in $\overline{G}$ forms the complete graph K_n. We claim that $(\Delta - \delta + 1)\delta < (n-1)(n-2)/2$, and hence

$$|E(G\overline{G}) - M| = \frac{n(n-1)}{2} > (n-1)^2 - \frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2}\,,$$

a contradiction, so that H cannot exist and $G\overline{G}$ must indeed be *Class 1*.

To prove the claim observe that $(\Delta - \delta + 1)\delta = -2\delta^2 + n\delta$ reaches its maximum for $\delta = n/4$, which satisfies $\delta \leq (n-2)/2$ as required by (2) (recall that $n \geq 4$). Ergo,

$$(\Delta - \delta + 1)\delta = (n - 2\delta)\delta \leq \left(n - \frac{2n}{4}\right)\frac{n}{4} = \frac{n^2}{8}\,.$$

Since $n^2/8 < (n-1)(n-2)/2$ if and only if $3n^2 - 12n + 8 > 0$, which is true for $n \geq 4$, the claim holds. □

3. No complementary prism can be subgraph-overfull

We have already proved that non-regular complementary prisms are all *Class 1*, thus not *SO*. Ergo, in order to prove that no complementary prism can be *SO*, it suffices to show the following.

Theorem 1. *No regular complementary prism is subgraph-overfull.*

Proof. Recall from Sect. 1 that if $G\overline{G}$ is a regular complementary prism, then G and $\overline{G}$ are d-regular graphs with $d = (n-1)/2$ and odd n. Since N is even, we know that $G\overline{G}$ cannot be overfull itself. Hence, if $G\overline{G}$ is *SO*, then it has a proper overfull subgraph H with $\Delta(H) = d + 1$.

Because a graph H is *overfull* if and only if $|V(H)|$ is odd and $\sum_{u\in V(H)}(\Delta(H) - d_H(u)) \leq \Delta(H) - 2$ [**6**], it suffices to show that every cut in $G\overline{G}$ has more than $d(G\overline{G}) - 2 = d - 1$ edges. Remark that this is actually stronger than showing that $G\overline{G}$ is not *SO*, since we prove this lower bound on the cardinality of every cut regardless of the parity of the cardinalities of the vertex sets separated by the cut.

For the sake of contradiction, let $Z \subsetneq V(G\overline{G})$ be such that $|\partial_{G\overline{G}}(Z)| \leq d - 1$, and let H be the subgraph of $G\overline{G}$ induced by Z. As in the proof of Lemma 1, let M be the perfect matching formed by the edges connecting the vertices of G and their corresponding vertices in $\overline{G}$. It is clear that H must contain at least one, but not all, edge(s) from M. Further, if $X := V(G) \cap Z$ and $Y := V(\overline{G}) \cap Z$, then

$$\partial_G(X) \cup \partial_{\overline{G}}(Y) \cup (M \cap \partial_{G\overline{G}}(Z)) = \partial_{G\overline{G}}(Z)\,.$$

Since $|\partial_{G\overline{G}}(H)| \leq d-1$ and it cannot be the case that $\partial_G(X)$ and $\partial_{\overline{G}}(Y)$ are both empty, we must have $|\partial_G(X)| \leq d - 2$ with $X \neq V(G)$, or $|\partial_{\overline{G}}(Y)| \leq d - 2$ with $Y \neq V(\overline{G})$. But this is a contradiction, since an n-vertex d-regular graph with odd n and $d \geq (n-1)/2$ cannot have a cut with fewer than $d - 1$ edges. □

4. All regular complementary prisms of order 18 are *Class 1*

In Table 1 we show the 5-edge-colourings which we have obtained for all of the 10 5-regular complementary prisms on 18 vertices. The vertices of each graph are represented by numbers from 1 to 18, being the first 9 vertices in the order from G and the last 9 vertices from $\overline{G}$. Each graph is represented by a 5-partition of its list of edges, each part of the partition corresponding to each of the 5 colour classes, numbered from 1 to 5. The list of edges is displayed also in 5 lines, each line for the perfect matching defined by the 9 edges of each colour class.

Table 1. 5-edge-colourings for all regular complementary prisms of order 18

Colour	Graph #1
1	(1, 2) (3, 4) (5, 8) (6, 7) (9, 18) (10, 15) (11, 16) (12, 17) (13, 14)
2	(1, 3) (2, 4) (5, 9) (6, 8) (7, 16) (10, 17) (11, 15) (12, 14) (13, 18)
3	(1, 4) (2, 3) (5, 14) (6, 9) (7, 8) (10, 16) (11, 17) (12, 18) (13, 15)
4	(1, 5) (2, 11) (3, 6) (4, 7) (8, 9) (10, 18) (12, 16) (13, 17) (14, 15)
5	(1, 10) (2, 5) (3, 12) (4, 13) (6, 15) (7, 9) (8, 17) (11, 18) (14, 16)
Colour	Graph #2
1	(1, 2) (3, 6) (4, 7) (5, 8) (9, 18) (10, 15) (11, 16) (12, 14) (13, 17)
2	(1, 3) (2, 4) (5, 9) (6, 8) (7, 16) (10, 18) (11, 17) (12, 13) (14, 15)
3	(1, 4) (2, 3) (5, 14) (6, 9) (7, 8) (10, 16) (11, 15) (12, 17) (13, 18)
4	(1, 5) (2, 11) (3, 7) (4, 6) (8, 9) (10, 17) (12, 18) (13, 14) (15, 16)
5	(1, 10) (2, 5) (3, 12) (4, 13) (6, 15) (7, 9) (8, 17) (11, 18) (14, 16)

Table 1. 5-edge-colourings for all regular complementary prisms of order 18 (cont.)

Colour	Graph #3
1	(1, 2) (3, 6) (4, 8) (5, 7) (9, 18) (10, 15) (11, 17) (12, 14) (13, 16)
2	(1, 3) (2, 4) (5, 9) (6, 8) (7, 16) (10, 17) (11, 15) (12, 18) (13, 14)
3	(1, 4) (2, 3) (5, 14) (6, 15) (7, 9) (8, 17) (10, 18) (11, 16) (12, 13)
4	(1, 5) (2, 11) (3, 7) (4, 6) (8, 9) (10, 16) (12, 17) (13, 18) (14, 15)
5	(1, 10) (2, 5) (3, 12) (4, 13) (6, 9) (7, 8) (11, 18) (14, 17) (15, 16)
Colour	Graph #4
1	(1, 2) (3, 4) (5, 6) (7, 9) (8, 17) (10, 15) (11, 16) (12, 18) (13, 14)
2	(1, 3) (2, 4) (5, 7) (6, 8) (9, 18) (10, 16) (11, 14) (12, 17) (13, 15)
3	(1, 4) (2, 3) (5, 14) (6, 9) (7, 8) (10, 17) (11, 18) (12, 15) (13, 16)
4	(1, 5) (2, 6) (3, 7) (4, 13) (8, 9) (10, 18) (11, 17) (12, 14) (15, 16)
5	(1, 10) (2, 11) (3, 12) (4, 8) (5, 9) (6, 15) (7, 16) (13, 18) (14, 17)
Colour	Graph #5
1	(1, 2) (3, 5) (4, 7) (6, 9) (8, 17) (10, 15) (11, 14) (12, 16) (13, 18)
2	(1, 3) (2, 4) (5, 7) (6, 15) (8, 9) (10, 16) (11, 17) (12, 18) (13, 14)
3	(1, 4) (2, 3) (5, 9) (6, 8) (7, 16) (10, 17) (11, 18) (12, 13) (14, 15)
4	(1, 5) (2, 6) (3, 12) (4, 8) (7, 9) (10, 18) (11, 16) (13, 15) (14, 17)
5	(1, 10) (2, 11) (3, 6) (4, 13) (5, 14) (7, 8) (9, 18) (12, 17) (15, 16)
Colour	Graph #6
1	(1, 2) (3, 5) (4, 6) (7, 9) (8, 17) (10, 15) (11, 14) (12, 18) (13, 16)
2	(1, 3) (2, 4) (5, 7) (6, 8) (9, 18) (10, 16) (11, 17) (12, 15) (13, 14)
3	(1, 4) (2, 3) (5, 14) (6, 9) (7, 8) (10, 17) (11, 18) (12, 13) (15, 16)
4	(1, 5) (2, 6) (3, 7) (4, 13) (8, 9) (10, 18) (11, 16) (12, 17) (14, 15)
5	(1, 10) (2, 11) (3, 12) (4, 8) (5, 9) (6, 15) (7, 16) (13, 18) (14, 17)
Colour	Graph #7
1	(1, 2) (3, 5) (4, 6) (7, 9) (8, 17) (10, 15) (11, 14) (12, 18) (13, 16)
2	(1, 3) (2, 4) (5, 8) (6, 7) (9, 18) (10, 16) (11, 17) (12, 15) (13, 14)
3	(1, 4) (2, 3) (5, 9) (6, 15) (7, 8) (10, 17) (11, 18) (12, 13) (14, 16)
4	(1, 5) (2, 6) (3, 7) (4, 13) (8, 9) (10, 18) (11, 16) (12, 17) (14, 15)
5	(1, 10) (2, 11) (3, 12) (4, 8) (5, 14) (6, 9) (7, 16) (13, 18) (15, 17)
Colour	Graph #8
1	(1, 2) (3, 5) (4, 7) (6, 9) (8, 17) (10, 15) (11, 16) (12, 18) (13, 14)
2	(1, 3) (2, 4) (5, 8) (6, 7) (9, 18) (10, 16) (11, 14) (12, 17) (13, 15)
3	(1, 4) (2, 3) (5, 9) (6, 8) (7, 16) (10, 17) (11, 18) (12, 13) (14, 15)
4	(1, 5) (2, 6) (3, 7) (4, 13) (8, 9) (10, 18) (11, 17) (12, 15) (14, 16)
5	(1, 10) (2, 11) (3, 12) (4, 8) (5, 14) (6, 15) (7, 9) (13, 18) (16, 17)
Colour	Graph #9
1	(1, 2) (3, 7) (4, 9) (5, 6) (8, 17) (10, 15) (11, 18) (12, 13) (14, 16)
2	(1, 3) (2, 4) (5, 8) (6, 15) (7, 9) (10, 16) (11, 14) (12, 18) (13, 17)
3	(1, 4) (2, 3) (5, 9) (6, 8) (7, 16) (10, 18) (11, 17) (12, 14) (13, 15)
4	(1, 5) (2, 6) (3, 8) (4, 7) (9, 18) (10, 17) (11, 16) (12, 15) (13, 14)
5	(1, 10) (2, 11) (3, 12) (4, 13) (5, 14) (6, 9) (7, 8) (15, 16) (17, 18)
Colour	Graph #10
1	(1, 2) (3, 8) (4, 5) (6, 15) (7, 9) (10, 16) (11, 17) (12, 14) (13, 18)
2	(1, 3) (2, 6) (4, 8) (5, 7) (9, 18) (10, 15) (11, 13) (12, 16) (14, 17)
3	(1, 4) (2, 3) (5, 9) (6, 8) (7, 16) (10, 17) (11, 18) (12, 13) (14, 15)
4	(1, 5) (2, 7) (3, 9) (4, 6) (8, 17) (10, 18) (11, 14) (12, 15) (13, 16)
5	(1, 10) (2, 11) (3, 12) (4, 13) (5, 14) (6, 7) (8, 9) (15, 18) (16, 17)

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