Acta Math. Univ. Comenianae
Vol. LXXXVIII, 3 (2019), pp. 507–510

BOUNDING THE COP NUMBER OF A GRAPH BY ITS GENUS

N. BOWLER, J. ERDE, F. LEHNER AND M. PITZ

ABSTRACT. The game of cops and robbers is a pursuit game played on a graph G in which a group of cops tries to catch a robber, where both are allowed to move along to edges of G. The cop number of G, denoted by $c(G)$, is the smallest number of cops needed to catch a robber on G. Schröder showed that $c(G) \leq \lfloor \frac{3}{2} \rfloor g(G) + 3$, where $g(G)$ is the genus of G, that is, the smallest k such that G can be drawn on an orientable surface of genus k. Furthermore, he conjectured that this bound could be improved to $c(G) \leq g(G) + 3$. By relating the game of cops and robbers to a topological game played on a surface we prove that $c(G) \leq \lceil \frac{4}{3} g(G) \rceil + 3$.

1. INTRODUCTION

The game of *cops and robbers* was introduced independently by Nowakowski and Winkler [**8**] and Quillot [**9**]. The game is a pursuit game played on a graph $G = (V, E)$ by two players, one player controlling a set of $k \geq 1$ *cops* and the other controlling a *robber*. Initially, the first player chooses a starting configuration $(c_1, c_2, \ldots, c_k) \in V^k$ for the cops and then the second player chooses a starting vertex $r \in V$ for the robber. Each *round* of the game consists of a move by the cops, and then a subsequent move by the robber. In each round a cop or robber can either move to a vertex adjacent to his current location, or stay still. Note that each cop may move in each round, and multiple cops may occupy the same vertex. The first player wins if at some time there is a cop on the same vertex as the robber, otherwise the robber wins. The *cop number* $c(G)$ of a graph G is the smallest number of cops k such that the first player has a winning strategy in this game.

Bounding $c(G)$ in terms of invariants of the graph G is a well studied problem (See for example [**4**]). Of particular interest is Meyniel's Conjecture, that for every graph G on n vertices $c(G) = O(\sqrt{n})$. Currently the best known bound is

$$c(G) = O\left(\frac{n}{2^{(1-o(1))\sqrt{\log n}}}\right)$$

proved independently by Frieze, Krivelevich and Loh [**5**], Lu and Peng [**6**], and Scott and Sudakov [**12**].

Received May 28, 2019.
2010 *Mathematics Subject Classification.* Primary 05C57.

In another direction, the cop number has been studied with regard to topological properties of the graph. As an early example of this Aigner and Fromme [**1**] showed that every planar graph has cop number at most three. For a graph G let us write $g(G)$ for the *genus* of G, that is, the smallest k such that G can be drawn on an orientable surface of genus k without crossing itself. Quillot [**10**] used similar ideas to show that, for every graph G, $c(G) \leq 2g(G) + 3$. These methods were refined by Schröder to give the following bound, which is currently the best known.

Theorem 1.1 (Schröder [**11**]). *For each graph G*

$$c(G) \leq \lfloor \frac{3}{2} g(G) \rfloor + 3.$$

These proofs all use the same basic idea as a tool, that a single cop can 'guard' a geodesic path in the graph G. Broadly, the strategy is then to inductively find collections of geodesic paths (initially in G, but later in some subgraph which the cops have restricted the robber to) such that, if we delete these paths, each component has strictly smaller genus than before. After a fixed number of steps the robber is restricted to a planar graph, in which three further cops can catch her by the result of Aigner and Fromme.

In Quillot's proof, two cops were needed to reduce the genus of the graph by one, and Schröder's improvement was to find a strategy such that three cops could be used to reduce the genus by two. Perhaps the best bound we could hope to prove with this strategy would be to reduce the genus of the graph by one using a single cop, and motivated by this, Schröder conjectured the following bound:

Conjecture 1.2 (Schröder [**11**]). For each graph G

$$c(G) \leq g(G) + 3.$$

However, Schröder also gave examples to show that this isn't always possible in a straightforward way, in particular he constructed a toroidal graph where every geodesic cycle was not null-homotopic. The main result of this paper will be to improve the bound in Theorem 1.1.

Theorem 1.3. *For each graph G*

$$c(G) \leq \lceil \frac{4}{3} g(G) \rceil + 4.$$

The only reference to lower bounds in the literature that we could find come from the survey paper of Bonato and Mohar [**3**] who give the following lower bound, which comes from a random construction of Mohar [**7**].

Theorem 1.4 (Mohar). *For each $g \in \mathbb{N}$ there exists a graph G such that*

$$c(G) \geq g(G)^{\frac{1}{2} - o(1)}.$$

In order to prove Theorem 1.3 we will introduce an auxiliary 'Waiter-Client' type topological game, and relate a winning strategy in this game to a winning strategy in the cops and robbers game.

Let S be a compact connected orientable surface with boundary. A *∂-arc* in S is either

- An arc A with endpoints x and y such that $A \cap \partial S = \{x, y\}$; or
- A closed curve A containing a point x such that $A \cap \partial S = \{x\}$.

Given a ∂-arc we will write ∂A for $\{x, y\}$ or $\{x\}$. Given a ∂-arc A in S we can form a new surface $\mathrm{Cut}(S, A)$ by 'cutting' S along A. Intuitively, cutting along A will result in two new arcs in the boundary corresponding to the two sides of A, which we will refer to as A and A'. We defer precise the definition of this operation to a later section.

The *Topological Marker-Cutter game* is a game played by two players, Marker and Cutter, on some compact connected orientable surface with boundary S with a distinguished point $x \in \partial S$ in the boundary of S. We let $S_0 = S$ and $X_0 = \{x\}$.

In a general turn we will have some 2-dimensional orientable surface with boundary S_k together with some set $X_k \subset \partial S_k$. First, Marker chooses two, not necessarily distinct, points x and y in X_k.

Then, Cutter chooses a ∂-arc $A_{k+1} \subset S_k$ such that $A_{k+1} \cap \partial S_k = \{x, y\}$. Cutter also chooses a component S_{k+1} of $\mathrm{Cut}(S_k, A_{k+1})$ and we set $X_{k+1} := (X \cup \{x', y'\}) \cap S_{k+1}$ where x' and y' are the new 'copies' of x and y in $\mathrm{Cut}(S_k, A_{k+1})$.

In this way a *play* of the game can be represented by a pair of sequences $(A_i, S_i \colon i \in \mathbb{N})$ of ∂-arcs and surfaces. Given a play of the game $(A_i, S_i \colon i \in \mathbb{N})$ the set of *active ∂-arcs* at turn k is

$$\mathcal{A}_k = \{A_i \colon A_i \subset \partial S_k\} \cup \{A'_i \colon A'_i \subset \partial S_k\},$$

the set of *active indices* is $I_k = \{i \colon A_i \in \mathcal{A}_k \text{ or } A'_i \in \mathcal{A}_k\}$ and we let $a_k = |I_k|$. Given a play the *score* of the game is $\sup_{i \in \mathbb{N}} a_i$, the maximum[1] number of active indices at any turn in the game.

We relate this game to the cops and robbers game played on a graph drawn on S by showing that, if S is a 2-dimensional orientable surface of genus g and G is a graph drawn on S, then $c(G) \leq v(S) + 1$.

Thus, to prove Theorem 1.3, it would be enough to show that $v(S) \leq \lceil \frac{4}{3} g \rceil + 3$. In fact, we show more, that this bound is tight up to a small additive constant.

Theorem 1.5. *Let S be a compact connected orientable surface with boundary of genus g. Then*

$$\lceil \frac{4}{3} g \rceil + 1 \leq v(S) \leq \lceil \frac{4}{3} g \rceil + 3.$$

The near-optimality of the bound in Theorem 1.5 suggests strongly that the bound of $\frac{4}{3} g(G) + 4$ in Theorem 1.3 is, up to an additive constant, the best bound that can be proven using a strategy of guarding geodesic paths.

References

1. Aigner M. and Fromme M., *A game of Cops and Robbers*, Discrete Appl. Math. **8** (1984), 1–12.
2. Andreae T., *On a pursuit game played on graphs for which a minor is excluded*, J. Combin. Theory Ser. B **41** (1986), 37–47.

[1]For a surface of finite genus it will be possible to bound the terms a_i and hence the score will be an attained maximum

3. Bonato A. and Mohar B., *Topological directions in Cops and Robbers*, `arXiv:1709.09050`.
4. Bonato A. and Nowakowski R., *The game of Cops and Robbers on graphs*, Stud. Math. Libr., vol. 61, AMS, Providence, 2011.
5. Frieze A., Michael K. and Loh P., *Variations on Cops and Robbers*, J. Graph Theory **69** (2012), 383–402.
6. Lu L. and Peng X., *On Meyniel's conjecture of the cop number*, J. Graph Theory **71** (2012), 192–205.
7. Mohar B., *Notes on Cops and Robber game on graphs*, Unpublished note (2008).
8. Nowakowski R. and Winkler P., *Vertex-to-vertex pursuit in a graph*, Discrete Math. **43** (1983), 235–239.
9. Quillot A., *Jeux et pointes xes sur les graphes*, Université de Paris VI (1978), 131–145.
10. Quilliot A., *A short note about pursuit games played on a graph with a given genus*, J. Combin. Theory Ser. B **38** (1985), 9–92.
11. Schröder B. S. W., *The copnumber of a graph is bounded by* $[3/2genus(g)] + 3$, Categorical perspectives (2001), 243–263.
12. Scott A. and Sudakov B., *A bound for the cops and robbers problem*, SIAM J. Discrete Math. **25** (2011), 1438–1442.

N. Bowler, Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany,
e-mail: `nathan.bowler@uni-hamburg.de`

J. Erde, Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany,
e-mail: `joshua.erde@uni-hamburg.de`

F. Lehner, Institute of Geometry, TU Graz, Graz, Austria,
e-mail: `f.lehner@tugraz.at`

M. Pitz, Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany,
e-mail: `max.pitz@uni-hamburg.de`