Acta Math. Univ. Comenianae
Vol. LXXXVIII, 3 (2019), pp. 561–566

TILING EDGE-COLOURED GRAPHS WITH FEW MONOCHROMATIC BOUNDED-DEGREE GRAPHS

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ABSTRACT. We prove that for all integers $\Delta, r \geq 2$, there is a constant $C = C(\Delta, r) > 0$ such that the following is true for every sequence $\mathcal{F} = \{F_1, F_2, \ldots\}$ of graphs with $v(F_n) = n$ and $\Delta(F_n) \leq \Delta$ for every $n \in \mathbb{N}$. In every r-edge-coloured K_n, there is a collection of at most C monochromatic copies from $\mathcal{F}$ whose vertex-sets partition $V(K_n)$. This makes progress on a conjecture of Grinshpun and Sárközy.

1. Introduction and main results

A conjecture of Lehel states that the vertices of any 2-edge-coloured complete graph can be partitioned into two monochromatic cycles of different colours. Here, single vertices and edges are considered cycles. This conjecture first appeared in [**2**], where it was also proved for some special types of colourings of K_n. Łuczak, Rödl and Szemerédi [**12**] proved Lehel's conjecture for all sufficiently large n using the regularity method. In [**1**], Allen gave an alternative proof, which gave a better bound on n. Finally, Bessy and Thomassé [**3**] proved Lehel's conjecture for all integers $n \geq 1$.

Moving on to more colours, Erdős, Gyárfás and Pyber [**6**] proved the following theorem.

Theorem 1 (Erdős–Gyárfás–Pyber, 1991). *The vertices of every r-edge-coloured complete graph on n vertices can be partitioned into $O(r^2 \log r)$ monochromatic cycles.*

It was further conjectured in [**6**] that r cycles should always suffice in Theorem 1. The currently best-known upper bound is due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [**9**], who showed that $O(r \log r)$ cycles suffice. The conjecture was refuted however by Pokrovskiy [**13**], who showed that, for every $r \geq 3$, there exist infinitely many r-coloured complete graphs which cannot be vertex-partitioned into r monochromatic cycles. Pokrovskiy conjectured though that in every r-coloured complete graph one can find r vertex-disjoint monochromatic cycles which cover all but at most c_r vertices for some $c_r \geq 1$ only depending on r (in his counterexample $c_r = 1$ is possible).

Received May 28, 2019.
2010 *Mathematics Subject Classification.* Primary 05C70; Secondary 05D10.
Key words and phrases. Tiling; vertex-partitioning.

In this paper, we study similar problems in which we are given a family of graphs $\mathcal{F}$ and an edge-coloured complete graph K_n and our goal is to partition $V(K_n)$ into monochromatic copies of graphs from $\mathcal{F}$. All families of graphs $\mathcal{F}$ we consider here are of the form $\mathcal{F} = \{F_1, F_2, \ldots\}$, where F_i is a graph on i vertices for every $i \in \mathbb{N}$ (note that the family of cycles is of this form since we consider vertices and edges to be cycles). We call such a family a *sequence of graphs.* An $\mathcal{F}$*-tiling* $\mathcal{T}$ of a graph G is a collection of vertex-disjoint copies of graphs from $\mathcal{F}$ in G with $V(G) = \bigcup_{T\in\mathcal{T}} V(T)$. If G is coloured, we say that $\mathcal{T}$ is *monochromatic* if every $T \in \mathcal{T}$ is monochromatic (but not necessarily in the same colour). Let $\tau_r(\mathcal{F}, n)$ be the minimum $t \in \mathbb{N}$ such that for every r-edge-coloured K_n, there is a monochromatic $\mathcal{F}$-tiling of size at most t. We call $\tau_r(\mathcal{F}) = \sup_{n\in\mathbb{N}} \tau_r(\mathcal{F}, n)$ the *tiling number* of $\mathcal{F}$.

Using this notation, the above results of Pokrovskiy and of Gyárfás, Ruszinkó, Sárközy and Szemerédi imply that $r+1 \le \tau_r(\mathcal{C}) \le O(r \log r)$, where $\mathcal{C}$ is the family of cycles. Note that, in general, it is not clear at all that $\tau_r(\mathcal{F})$ is finite and it is a natural question to ask for which families this is the case.

The study of such tiling problems for more general families of graphs was initiated by Grinshpun and Sárközy [**8**] who proved the following result. The *maximum degree* $\Delta(\mathcal{F})$ of a sequence of graphs $\mathcal{F}$ is given by $\max_{F\in\mathcal{F}} \Delta(F)$, where $\Delta(F)$ is the maximum degree of F.

Theorem 2 (Grinshpun–Sárközy [**8**], 2016)**.** *Let $\mathcal{F}$ be a sequence of graphs of maximum degree Δ. Then, we have*

$$\tau_2(\mathcal{F}) \le 2^{O(\Delta \log \Delta)}.$$

In particular, $\tau_2(\mathcal{F})$ is finite whenever $\Delta(\mathcal{F})$ is finite.

Grinshpun and Sárközy also proved that $\tau_2(\mathcal{F}) \le 2^{O(\Delta)}$ for every sequence of bipartite graphs $\mathcal{F}$ of maximum degree Δ and showed that this is best possible up to the implicit constant (using a result of Graham, Rödl and Ruciński [**7**]). Sárközy [**14**] further proved that Theorem 2 can be improved a lot for the special case of powers of cycles (the k-th power of a graph H is the graph obtained from H by adding an edge between any two vertices of distance at most k).

For more than two colours less is known. Answering a question of Elekes, Soukup, Soukup and Szentmiklóssy [**5**], Bustamante, Frankl, Pokrovskiy, Skokan and the first author [**4**] proved that $\tau_r(\mathcal{C}^{(k)}) < \infty$ for all $r, k \in \mathbb{N}$, where $\mathcal{C}^{(k)}$ is the family of k-th powers of cycles. Grinshpun and Sárközy [**8**] conjectured that the same should be true for all families of graphs of bounded degree (with a much stronger bound).

Conjecture 1 (Grinshpun–Sárközy [**8**], 2016)**.** For every $r \in \mathbb{N}$, there is some $C_r \in \mathbb{N}$ so that for every sequence $\mathcal{F}$ of graphs of maximum degree Δ, we have $\tau_r(\mathcal{F}) \le 2^{\Delta^{C_r}}$.

Note that it is still open to decide whether $\tau_r(\mathcal{F})$ is finite for such families. We show that this is the case and make progress towards Conjecture 1.

Theorem 3. *For all integers $r, \Delta \geq 2$ and all families of graphs $\mathcal{F}$ of maximum degree Δ, we have $\tau_r(\mathcal{F}) \leq r^{r^{O(\Delta^5)}}$. In particular, $\tau_r(\mathcal{F}) < \infty$ whenever $\Delta(\mathcal{F}) < \infty$.*

1.1. Graphs with linear Ramsey number

Given a sequence of graphs $\mathcal{F}$, it follows from the pigeon-hole principle that every r-edge-coloured K_n contains a monochromatic copy from $\mathcal{F}$ of size at least $n/\tau_r(\mathcal{F})$. Finding the largest such copy is closely related to Ramsey numbers. The r-colour Ramsey number $R_r(G)$ of a graph G is the smallest integer N such that every r-coloured K_N contains a monochromatic copy of G. We say that $\mathcal{F} = \{F_1, F_2, \ldots\}$ has linear Ramsey number if $R_r(F_n) = O(n)$. If $\mathcal{F}$ is increasing, i.e. $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ for every $i \in \mathbb{N}$, then $\mathcal{F}$ has linear Ramsey number if and only if there is some $c > 0$ such that every r-edge-coloured K_n contains a monochromatic copy from $\mathcal{F}$ of size at least cn (the only if part is always true). Hence, having linear Ramsey number is a necessary condition for having finite tiling number in this case. We make the following conjecture.

Conjecture 2. Every sequence of graphs $\mathcal{F}$ with linear r-colour Ramsey number has finite r-colour tiling number.

Conjecture 2 is probably very difficult. A recent breakthrough result of Lee [**11**] asserts that graphs with bounded degeneracy have linear Ramsey number. It would be of great interest to prove Conjecture 2 for these graphs.

Graphs with linear Ramsey number are well studied and we can use the results from this area to obtain lower bounds on the tiling number: It was proved by Graham, Rödl and Ruciński [**7**] that there exists a sequence of bipartite graphs $\mathcal{F} = \{F_1, F_2, \ldots\}$ with $R_2(F_n) \geq 2^{\Omega(\Delta)}n$. Grinshpun and Sárközy observed that it is easy to make this sequence increasing, thereby showing that $\tau_2(\mathcal{F}) \geq 2^{\Omega(\Delta)}$ as well.

2. Proof overview

We will use the absorption method introduced by Erdős, Gyárfás and Pyber in [**6**]. This method has become a standard tool and has been applied to many problems in the area. We will briefly sketch the proof of Theorem 1 in order to introduce the method and then explain how we need to adapt it for our problem. For sake of clarity, we will not make an effort to calculate the exact constants and use the following standard O-notation. Given two functions $f, g : \mathbb{N} \to \mathbb{N}$ and a constant $C > 0$, we say that $f = O_C(g)$ if there is a constant $C' = C'(C) > 0$ so that $f(n) \leq C' \cdot g(n)$ for all but finitley many $n \in \mathbb{N}$. We say that $f = \Omega_C(g)$ if $g = O_C(f)$. For example, Theorem 1 implies that $\tau_r(\mathcal{C}) = O_r(1)$.

Proof Sketch of Theorem 1. We start by defining absorbers for the family of cycles, which will play a central role in the proof.

Definition 1. A pair (H, A) of a graph H and a set $A \subset V(H)$ is called an *absorber* if $H[V(H) \smallsetminus X]$ contains a Hamilton cycle for every $X \subset A$.

Fix $r, n \in \mathbb{N}$ and an r-edge-coloured K_n now. The first part of the proof is finding a large monochromatic absorber.

Lemma 1. *There is a monochromatic absorber (H, A) with $|A| \geq \Omega_r(n)$.*

Then, we greedily cover most of the vertices by repeatedly taking out the largest monochromatic cycle.

Lemma 2. *There is a collection of $O_r(1)$ vertex-disjoint monochromatic cycles in $V(K_n) \setminus V(H)$, covering all but $\varepsilon(|A|)$ vertices, where $\varepsilon > 0$ is a small constant depending on r.*

The key part of the proof is to deal with the set R of leftover vertices. This is done using the following Absorption Lemma.

Lemma 3 (Absorption Lemma for cycles)**.** *Let V_1, V_2 be sets with $|V_1| \leq \varepsilon |V_2|$ and let G be an r-coloured complete bipartite graph with parts V_1, V_2. Then, there is a collection of $O_r(1)$ vertex-disjoint monochromatic cycles in G covering V_1.*

In order to finish the proof, we apply the Absorption Lemma to the complete bipartite graph induced by $V_1 = R$ and $V_2 = A$ and denote by X the set of vertices in A which are covered in this process. Finally, using the property of the absorber H, we find a monochromatic cycle whose vertex-set is $V(H) \setminus X$. □

In order to prove Theorem 3, we will follow the basic strategy explained above. We will use so called "super-regular pairs" as absorbers, combined with the blow-up lemma [**10**] which guarantees a similar property as in Definition 1. The process of adapting Lemma 1 and Lemma 2 to our problem is done in a similar way as in [**8**] and therefore we will not discuss this here.

The main difficulty of the proof is the following absorption lemma. For simplicity we will describe it only for families of bipartite graphs.

Lemma 4 (Absorption Lemma for graphs of bounded degree)**.** *Fix integers $\Delta, r \geq 2$ and a sequence of bipartite graphs $\mathcal{F}$ with $\Delta(\mathcal{F}) \leq \Delta$. Let V_1, V_2 be disjoint sets with $|V_1| \leq |V_2|$ and let G be the graph obtained from $K(V_1, V_2)$ (the complete bipartite graph with parts V_1 and V_2) by replacing V_1 with a clique. Suppose that G is edge-coloured with r colours. Then, there is a collection of $O_{\Delta,r}(1)$ vertex-disjoint monochromatic copies from $\mathcal{F}$ in G which cover V_1.*

The proof of Lemma 4 proceeds by induction on r. In order to apply the induction hypothesis, we will prove a stronger statement which does not require the host graph to be complete but instead that $\deg(v, V_2) \geq \delta |V_2|$ for all $v \in V_1$. The basic steps of the proof can be summarised as follows.

(i) Find a monochromatic, say red, absorber with vertices $U_1 \cup U_2$, where $U_i \subset V_i$ for $i = 1, 2$.
(ii) Greedily cover most of $V_1 \setminus U_1$ and let $R \subset V_1$ be the set of leftover vertices.
(iii) Partition $R = R_1 \cup R_2 \cup R_3$ in such a way that the vertices in R_1 have many red neighbours in U_2, the vertices in R_2 have many non-red neighbours in U_2, and the vertices in R_3 don't have many neighbours in U_2. The vertices in R_1

can then be included in the absorber and the vertices in R_2 can be dealt with by induction. Since, the vertices in R_3 don't have many neighbours in U_2, we will have $\deg(v, V_2) \geq (\delta + \delta') \, |V_2 \setminus U_2|$ for some small constant $\delta' = \delta'(\Delta, r)$. We iterate this process now on the graph induced on $R_3 \cup V_2 \setminus U_2$. Since $\deg(v, V_2)$ cannot be larger than $|V_2|$, this process will end after $O_{\Delta,r}(1)$ iterations.

Proving Theorem 3 for families of bipartite graphs using Lemma 4 is straightforward. However, for families of non-bipartite graphs, things get more complicated. Clearly, we cannot hope for an absorption lemma like Lemma 3 as the host graph is bipartite and thus all its subgraphs are. Instead, we will work in a multi-partite graph. We then prove a variation of Lemma 4 in which G has $\Delta + 1$ parts and we require that every vertex in V_1 is in many monochromatic cliques with one vertex in each part. However, when applying this absorption lemma, we can't guarantee that this property holds (consider, for example, a three-partite graph in which the edges between V_1 and V_2 are blue and all other edges are red). We will overcome this problem by iteratively applying the absorption lemma and prove that, after $O_{\Delta,r}(1)$ iterations, all vertices are covered. This idea was recently introduced in [**4**].

Acknowledgment. This project began during the 'Rio Workshop on Extremal and Structural Combinatorics" held at IMPA, Rio de Janeiro, Brazil in January 2018. We thank the organisers of the workshop and IMPA for the stimulating working environment.

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