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THE STRUCTURE OF HYPERGRAPHS WITHOUT LONG BERGE CYCLES

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Abstract. We study the structure of r-uniform hypergraphs containing no Berge cycles of length at least k for $k \leq r$, and determine that such hypergraphs have some special substructure. In particular we determine the extremal number of such hypergraphs, giving an affirmative answer to the conjectured value when $k = r$ and giving a a simple solution to a recent result of Kostochka-Luo when $k < r$.

1. Introduction

In 1959 Erdős and Gallai proved the following results on the Turán number of paths and families of long cycles.

Theorem 1 (Erdős, Gallai [**2**]). *Let $n \geq k \geq 1$. If G is an n-vertex graph that does not contain a path of length k, then $e(G) \leq \frac{(k-1)n}{2}$.*

Theorem 2 (Erdős, Gallai [**2**]). *Let $n \geq k \geq 3$. If G is an n-vertex graph that does not contain a cycle of length at least k, then $e(G) \leq \frac{(k-1)(n-1)}{2}$.*

In fact, Theorem 1 was deduced as a simple corollary of Theorem 2. Recently numerous mathematicians started investigating similar problems for r-uniform hypergraphs. We will refer to r-uniform hypergraphs as an r-graphs for simplicity. All r-graphs are simple (i.e. contain no multiple edges), unless stated otherwise.

Definition 3. A *Berge cycle* of length t in a hypergraph, is an alternating sequence of distinct vertices and hyperedges, $v_0, e_1, v_1, e_2, v_2, \dots, v_{t-1}, e_t, v_0$ such that, $v_{i-1}, v_i \in e_i$, for $i = 1, 2, \dots t$, (where indices taken modulo t).

Definition 4. A *Berge path* of length t in a hypergraph, is an alternating sequence of distinct vertices and hyperedges, $v_0, e_1, v_1, e_2, v_2, \dots, e_t, v_t$ such that, $v_{i-1}, v_i \in e_i$, for $i = 1, 2, \dots t$.

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The first extension of Erdős and Gallai [**2**] result, was by Győri, Katona, and Lemons [**6**], who extended Theorem 1 for r-graphs. It turns out that the extremal numbers have a different behavior when $k \leq r$ and $k > r$

Theorem 5 (Győri, Katona and Lemons [**6**])**.** *Let $r \geq k \geq 3$, and let $\mathcal{H}$ be an n-vertex r-graph with no Berge path of length k. Then $e(\mathcal{H}) \leq \frac{(k-1)n}{r+1}$.*

Theorem 6 (Győri, Katona and Lemons [**6**])**.** *Let $k > r + 1 > 3$, and let $\mathcal{H}$ be an n-vertex r-graph with no Berge-path of length k. Then $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r}$.*

The remaining case when $k = r + 1$ was solved later by Davoodi, Győri, Methuku, and Tompkins [**1**], the extremal number matches the upper bound of Theorem 6.

Similarly the extremal hypergraphs when Berge cycles of length at least k are forbidden, are different in the cases when $k \geq r+2$ and $k \leq r+1$ with an exceptional third case when $k = r$. The latter has a surprisingly different extremal hypergraph. Fűredi, Kostochka and Luo [**4**] provide sharp bounds and extremal constructions for infinitely many n, for $k \geq r + 3 \geq 6$. Later they [**5**] also determined exact bounds and extremal constructions for all n, for the case $k \geq r + 4$. Kostochka and Luo [**10**] determine a bound for $k \leq r - 1$ which is sharp for infinitely many n. Ergemlidze, Győry, Metukhu, Salia, Tompkins and Zamora [**3**] determine a bound in the cases where $k \in \{r + 1, r + 2\}$. The case when $k = r$ remained open. Both papers [**10, 3**] conjectured the maximum number of edges to be bounded by $\max\left\{\frac{(n-1)(r-1)}{r}, n - (r - 1)\right\}$ (See figure 14).

Theorem 7 (Füredi, Kostochka and Luo [**4, 5**])**.** *Let $r \geq 3$ and $k \geq r+3$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length k or longer. Then $e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$.*

Theorem 8 (Ergemlidze et al. [**3**])**.** *If $k \geq 4$ and $\mathcal{H}$ is an n-vertex r-graph with no Berge cycles of length at least k, then if $k = r + 1$ then $e(\mathcal{H}) \leq n - 1$, and if $k = r + 2$ then $e(\mathcal{H}) \leq \frac{(n-1)(r+1)}{r}$.*

Theorem 9 (Kostochka, Luo [**10**])**.** *Let $k \geq 4, r \geq k + 1$ and let $\mathcal{H}$ be an n-vertex r-uniform multi-hypergraph, each edge of $\mathcal{H}$ has multiplicity at most $k - 2$. If $\mathcal{H}$ has no Berge-cycles of length at least k, then $e(H) \leq \frac{(k-1)(n-1)}{r}$.*

Kostochka and Luo obtain their result from the incidence bipartite graph by investigating the structure of 2-connected bipartite graphs. In a similar way a previous result of Jackson [**9**] gives an upper bound on the number of edges of a multi r-graph with no Berge cycle of length at least r.

Theorem 10 (Jackson [**9**])**.** *Let G be a bipartite graph with bi-partition A and B such that $|A| = n$ and every vertex in B has degree at least r, if $|B| > \left\lfloor\frac{n-1}{r-1}\right\rfloor(r-1)$ then G contains a cycle of length at least $2r$.*

In this paper we study the structure of r-graphs containing no Berge cycles of length at least k, for all $3 \leq k \leq r$. By exploring the structure of the hypergraphs, instead of bipartite graphs, we are able to find extremal number in the case when $k = r$, which also gives us a simple proof for Theorem 9. Even more our method lets us determine the extremal number for every value of n in both simple r-graphs and multi r-graphs.

2. Notation and results

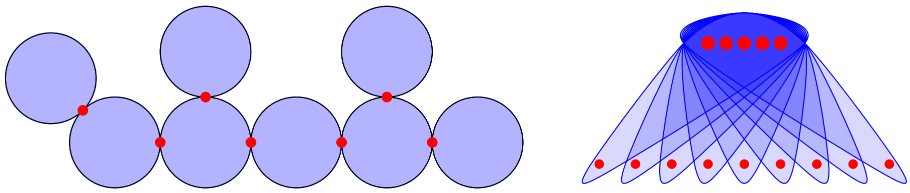

Figure 1. The extremal graphs from Theorems 12, 14 and 16. The figure on the left is a block tree, each block contains same number of vertices, either r in the case of multi-hypergraphs or $r+1$ otherwise and $k-1$ hyperedges. The figure on the right is $\mathcal{S}_n^{(r)}$ the n-vertex r-star, each hyperedge share the same $r-1$ vertices.

Given a hypergraph $\mathcal{H}$, let $V(\mathcal{H})$ and $E(\mathcal{H})$ denote the set of vertices and hyperedges of $\mathcal{H}$, respectively, and let $v(\mathcal{H}) := |V(\mathcal{H})|$, $e(\mathcal{H}) := |E(\mathcal{H})|$. We denote by $\mathbb{1}_{r\mathbb{N}^*}(n)$, the characteristic function of $r\mathbb{N}^*$, the function which is 1 when n is a positive multiple of r and 0 otherwise. A hypergraph is $\mathcal{F}$-free if it doesn't contain a copy of any hypergraph from the family $\mathcal{F}$ as a sub-hypergraph. In the following, we are particularly interested in the families $\mathcal{BP}_k$ and $\mathcal{BC}_{\geq k}$, the family of Berge path of length k and the family of Berge cycles of length at least k, respectively. The Turán number $\mathrm{ex}_r(n, \mathcal{F})$ and $\mathrm{ex}_r^{\mathrm{multi}}(n, \mathcal{F})$ are the maximum number of hyperedges in a $\mathcal{F}$-free hypergraph or multi-hypergraph respectively on n vertices.

Let $\mathcal{H}$ be a hypergraph. Then its 2-*shadow*, denoted by $\partial_2\mathcal{H}$, is the collection of pairs of vertices that lie in some hyperedge of $\mathcal{H}$. The graph $\mathcal{H}$ is *connected* if $\partial_2(\mathcal{H})$ is a connected graph.

Let n, k, r be integers such that $k \leq r$, for fix $s \in \{r, r+1\}$. A r-graph $\mathcal{H}$ is called a $(s, k-1)$-block tree if $\partial_2(\mathcal{H})$ is connected and every 2-connected block of $\partial_2(\mathcal{H})$ consists of s vertices which induce $k-1$ hyperedges in $\mathcal{H}$. A $(s, k-1)$-block tree contains no Berge-cycle of length at least k, because each of its blocks contain fewer than k hyperedges.

We define the r-star, $\mathcal{S}_n^{(r)}$, as the n-vertex r-graph with vertex set $V(\mathcal{S}_n^{(r)}) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(\mathcal{S}_n^{(r)}) = \{\{v_1, v_2, \ldots, v_{r-1}, v_i\} : r \leq i \leq n\}$, the set $\{v_1, v_2, \ldots, v_{r-1}\}$ is called the center of the star. Since $\mathcal{S}_n^{(r)}$ has just $r-1$ vertices of degree bigger than 1, then $\mathcal{S}_n^{(r)}$ contains no Berge cycle of length at least r.

Definition 11. For a set $S \subseteq V$, the *hyperedge neighborhood* of S in a r-graph $\mathcal{H}$ is the set

$$N_h(S) := \{h \in E(\mathcal{H}) | h \cap \mathcal{S} \neq \emptyset\}$$

of hyperedges that are incident with at least one vertex of S.

Our Main results are:

Theorem 12. *Let k, n and r be positive integers such that $4 \leq k < r$, then*

$$\mathrm{ex}_r(n, \mathcal{BC}_{\geq k}) = \left\lfloor \frac{n-1}{r} \right\rfloor (k-1) + \mathbb{1}_{r\mathbb{N}^*}(n)$$

If $r|(n-1)$ the only extremal n-vertex r-graphs are the $(r+1, k-1)$-block trees.

We note that as a corollary of Theorem 12 we obtain a slightly stronger version of Theorem 5

Corollary 13. *Let k, n and r be positive integer with $4 \leq k \leq r$, then*

$$\mathrm{ex}_r(n, \mathcal{BP}_k) = \left\lfloor \frac{n}{r+1} \right\rfloor (k-1) + \mathbb{1}_{(r+1)\mathbb{N}^*}(n+1)$$

Theorem 14. *Let $r > 2$ and n be positive integers, then*

$$\mathrm{ex}_r(n, \mathcal{BC}_{\geq r}) = \max\left\{ \left\lfloor \frac{n-1}{r} \right\rfloor (r-1), n-r+1 \right\}$$

When $n - r + 1 > \frac{n-1}{r}(r-1)$ the only extremal graph is $\mathcal{S}_n^{(r)}$. When $\frac{n-1}{r}(r-1) > n - r + 1$ and $r|(n-1)$ the only extremal graphs are the $(r+1, k-1)$-block trees.

Remark 15. *In particular when $n \geq r(r-2)+2$, we have that $\mathrm{ex}_r(n, \mathcal{BC}_{\geq r}) = n - r + 1$ and $\mathcal{S}_n^{(r)}$ is the only extremal hypergraph.*

Theorem 16. *Let k, n and r be positive integers such that $2 \leq k \leq r$. Then*

$$\mathrm{ex}_r^{\mathrm{multi}}(n, \mathcal{BC}_{\geq k}) = \left\lfloor \frac{n-1}{r-1} \right\rfloor (k-1)$$

If $r-1|(n-1)$ the only extremal graphs with n vertices are the $(r, k-1)$-block trees.

As a corollary of Theorem 16 we obtain a version of Theorem 5 with multiple hyperedges

Corollary 17. *Let k, n and r be positive integer with $2 \leq k \leq r$ then*

$$\mathrm{ex}_r^{\mathrm{multi}}(n, \mathcal{BP}_k) = \left\lfloor \frac{n}{r} \right\rfloor (k-1).$$

In fact all these results have essentially the same proof since, these results follow from our Lemma 18, which to some extent lets us understand the structure of long Berge cycle free hypergraphs.

Lemma 18. *Let r, n and m be positive integers, with $n > r$, and let $\mathcal{H}$ be an n-vertex r-graph which is $\mathcal{BC}_{\geq k}$-free such that every hyperedge has multiplicity at most m. Then at least one of the following holds.*

i) *There exists* $S \subseteq V$ *of size* $r-1$ *such that* $|N_h(S)| \leq m$. *Moreover, if* $m < k-1$ *there exists a set* S *of size* $r-1$ *such that* $N_h(S)$ *is* $d \leq m$ *copies of a hyperedge* h *and* $S \subset h$.

ii) *There exists* $S \subseteq V$ *of size* r *such that* $|N_h(S)| \leq k-1$.

iii) $k = r$, $m < k-1$, *and there exists* $e \in E(\mathcal{H})$ *such that after removing* e *from* $\mathcal{H}$ *the resulting* r*-graph can be decomposed in two* r*-graphs,* $\mathcal{S}$ *and* $\mathcal{K}$ *sharing one vertex, such that* $\mathcal{S}$ *is a* r*-star with at least* $r-1$ *edges, the shared vertex is in the center of* $\mathcal{S}$, $e \cap V(\mathcal{S})$ *is a subset of the center of* $\mathcal{S}$ *and* $v(\mathcal{K}) \geq 2$.

In particular, since no hyperedge can have multiplicity larger than $k-1$, *by setting* $m = k-1$ *we have that there exists a set* S *of size* $r-1$ *incident with at most* $k-1$ *edges.*

In the manuscript [**7**], we prove Theorems 12, 14 and 16 including Lemma 18, as well as their corollaries.

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