Acta Math. Univ. Comenianae
Vol. LXXXVIII, 3 (2019), pp. 885–890

ENUMERATION OF UNSENSED ORIENTABLE AND NON-ORIENTABLE MAPS

E. KRASKO AND A. OMELCHENKO

ABSTRACT. The work is devoted to the problem of enumerating maps on an orientable or non-orientable surface of a given genus g up to all symmetries (so called unsensed maps). We obtain general formulas which reduce the problem of counting such maps to the problem of enumerating rooted quotient maps on orbifolds. In addition, we solve the problem of describing all cyclic orbifolds for a given orientable or non-orientable surface of fixed genus g. We also derive recurrence relations for quotient rooted maps on orbifolds that can be orientable or non-orientable surfaces with r branch points, h boundary components and g handles or cross-caps. These results allowed us to calculate the numbers of unsensed maps on orientable or non-orientable surfaces of arbitrary genus g by the number of edges.

By a topological map M on a surface X we will mean a 2-cell imbedding of a connected graph G, loops and multiple edges allowed, into a compact connected 2-dimensional manifold X without boundary, such that the connected components of $X - G$ are 2-cells. The 0-, 1-, and 2-dimensional cells of a map M are its vertices, edges, and faces, respectively [**11**]. In this paper we consider both orientable and non-orientable surfaces without boundary. Every such surface can be characterized by its genus g. The orientable surface X of genus g is a sphere with g handles. The non-orientable surface of genus g is a sphere with g holes glued with crosscaps (or Möbius bands). Sometimes instead of g we will use the Euler characteristic χ of the surface X, equal to $2-2g$ in the case of an orientable surface X^+ and $2-g$ in the case of a non-orientable surface X^-.

Two topological maps M_1 and M_2 on a surface X are said to be isomorphic if there is a homeomorphism h of X that induces an isomorphism of the underlying graphs G_1 and G_2. Map isomorphism splits the set of all maps on X into equivalence classes, and each such class is called an unlabelled map. For orientable surfaces we have two types of homeomorphisms, orientation-preserving and orientation-reversing. Unlabelled maps on an orientable surface X^+ up to only orientation-preserving homeomorphisms are called sensed maps. Unlabelled maps on an orientable or a non-orientable surface up to all homeomorphisms are

Received May 30, 2019.
2010 *Mathematics Subject Classification.* Primary 05C30, 05C10, 05A15.
Key words and phrases. Map; surface; orbifold; enumeration; sensed maps; unsensed maps.
The publication was prepared within the framework of the Academic Fund Program at the National Research University Higher School of Economics (HSE) in 2019-2020 (grant No 19-01-004) and by the Russian Academic Excellence Project “5-100”.

called unsensed maps. The main aim of our work is to enumerate unsensed maps on both orientable and non-orientable surfaces.

A general technique for counting sensed planar maps on the sphere was developed by Liskovets [**19**] in the early eighties. His approach reduces the enumerating problem for sensed maps on the sphere to counting *quotient maps on orbifolds*, maps on quotients of a surface under a finite group of automorphisms. Mednykh and Nedela [**21**] significantly developed Liskovets' approach, obtaining general formulas for counting sensed maps on an arbitrary orientable surface of *a given genus* g. The most important result of that paper was an explicit analytical formula for integer coefficients, that have the meaning of the numbers of epimorphisms from fundamental groups $\pi_1(O)$ of orbifolds to cyclic groups Z_d. It was proved that having these coefficients one can reduce the combinatorial problem of enumerating sensed maps on a surface of a given genus g to the enumeration of rooted quotient maps on orbifolds. The latter problem can be solved, for example, via the recursive approach developed by Tutte [**23**]. As a result appeared a whole series of papers devoted to enumerating various kinds of sensed maps. In particular, regular sensed maps on the torus [**17**] were enumerated, as well as regular sensed maps on orientable surfaces of a given genus g [**16**], sensed hypermaps [**22**], one-face regular sensed maps [**15**] and so on.

The current progress in enumeration of unsensed maps is not as good as for sensed maps. There exist only a few results [**3**], [**20**], [**18**], [**14**] devoted to enumeration of unsensed maps on orientable surfaces, and all of them significantly use some specifics of the enumerated objects. For instance, enumeration regardless of genus [**3**] does not require listing orbifolds for each genus individually. For small genera surfaces, such as the sphere [**20**] and the torus [**18**], the corresponding orbifolds are quite simple and can be explicitly described. The corresponding problem for unsensed maps on non-orientable surfaces seem to have never been considered at all.

In the first part of our work we present two general formulas that allow to compute the number of maps up to all symmetries on surfaces of a given genus g. For orientable surfaces we have

$$(1)\quad \bar{\tau}_{X_\chi^+}(n) = \frac{1}{2}\Big(\tilde{\tau}_{X_g^+}(n) + \frac{1}{2n}\sum_{\substack{m|2n\\ lm=n}}\ \sum_{O\in Orb^-(X_\chi^+/Z_{2l})} \tau_O(2m)\cdot Epi_o^+(\pi_1(O), Z_{2l})\Big),$$

for non-orientable we have

$$(2)\quad \bar{\tau}_{X_\chi^-}(n) = \frac{1}{4n}\sum_{\substack{m|2n\\ lm=2n}}\ \sum_{O\in Orb(X_\chi^-/Z_l)} \tau_O(2m)\cdot (Epi_o(\pi_1(O), Z_l) - Epi_o^+(\pi_1(O), Z_l)).$$

These formulas express the number of unsensed maps on any such surface as a linear combination of numbers of quotient maps on cyclic orbifolds with integer coefficients. Moreover, similarly to the results obtained in [**21**], we express these coefficients through the numbers of epimorphisms from fundamental groups $\pi_1(O)$ of orbifolds to cyclic groups Z_d and obtain exact analytical formulas for them.

It is worth noting that with these formulas we also solve a problem important for algebraic topology: describing all cyclic coverings with the covering surface being a given orientable or non-orientable surface of genus g. Previously, this description was available only for coverings that arise from orientation-preserving homeomorphisms, in [**21**]. The list of new orbifolds obtained by us can be found for small values of g in Table 1. For arbitrary g we implemented a program available in open access. It allows anyone to obtain the corresponding list of orbifolds for a given orientable or non-orientable surface.

Using the results obtained in the first part of our paper we reduce the problem of enumerating unsensed maps to the problem of enumerating quotient maps on orbifolds of a certain form. It is worth noting that this result makes it possible not only to solve the problem of enumerating all unsensed maps on surfaces of a given genus g, but also enumerate other objects of a similar kind, such as hypermaps, d-regular maps, one-face maps, etc.

The second part of the present work is devoted to the construction of recurrence relations for counting rooted quotient maps on orbifolds, the list of which was obtained in the first part. To derive these relations, a standard technique known as recursive approach [**23**] is used. It turns out, however, that the structure of quotient maps on cyclic orbifolds in the general case is more complex than for small genus surfaces. For instance, to enumerate quotient maps on the disc, one of the quotients of a sphere, one can reduce this problem to enumerating quadrangulations [**20**] due to the fact that every planar quadrangulation is bipartite, which is not true for higher genera surfaces. Orbifolds that arise from orientation-reversing homeomorphisms of the torus have no branch points [**18**], which also means that they are easier to enumerate. In the general case of unsensed maps of a given genus, we have to consider orbifolds that can be surfaces with r branch points, h boundary components and g handles or cross-caps. Recurrence relations for the numbers of quotient maps on such orbifolds depend on a large number of additional parameters, which significantly complicates the construction of such recurrence relations. Despite that, we were able to construct recurrence relations for an arbitrary orientable or non-orientable orbifold of the form described above. For some simple cases (the disk, the Möbius band) we provide and prove them explicitly. Explicitly writing out such recurrence relations for more complex orbifolds does not present a separate interest, would only be technically complex and result in cumbersome expressions. Instead, we decided to describe in detail the main steps in constructing these recurrence relations, and implement an algorithm realizing the corresponding computations. Our program written in Go is available online for orientable and for non-orientable orbifolds. It can enumerate quotient maps on the above-described orbifolds and obtain the final results for unsensed maps.

The recurrence relations constructed in the second part of the paper together with the formulas obtained in the first part completely solve the problem of enumerating all unsensed maps on orientable and non-orientable surfaces. In the presented work we provide the numbers of such maps enumerated by the number of edges on orientable surfaces of genus up to 10 and on non-orientable surfaces

Table 1. Orbifolds covered by Euler characteristic X surfaces. For $X \geq 0$ only the first few rows are listed. For even X all $Epi_o(\pi_1(O), Z_l)$ epimorphisms are split into those corresponding to coverings by all and only by orientable surfaces of Euler characteristic X.

X	l	$\pm$	χ	h	m_i	$Epi_o(\pi_1(O), Z_l)$	X	l	$\pm$	χ	h	m_i	$Epi_o(\pi_1(O), Z_l)$	$Epi_o^+(\ldots)$
1	1	−	1	0	[]	1	0	1	−	0	0	[]	1	0
1	2	+	1	1	[2]	1	0	2	+	1	1	[2 2]	1	0
1	3	−	1	0	[3]	2	0	2	−	1	0	[2 2]	2	0
1	4	+	1	1	[4]	2	0	2	+	0	2	[]	2	1
1	5	−	1	0	[5]	4	0	2	−	0	0	[]	3	1
1	6	+	1	1	[6]	2	0	2	−	0	1	[]	2	1
1	7	−	1	0	[7]	6	0	3	−	0	0	[]	2	0
1	8	+	1	1	[8]	4	0	4	+	0	2	[]	2	0
1	9	−	1	0	[9]	6	0	4	−	0	0	[]	4	4
1	10	+	1	1	[10]	4	0	4	−	0	1	[]	2	0
1	11	−	1	0	[11]	10	0	5	−	0	0	[]	4	0
1	12	+	1	1	[12]	4	0	6	+	0	2	[]	4	2
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
−1	1	−	−1	0	[]	1	−2	1	−	−2	0	[]	1	0
−1	2	+	1	1	[2 2 2]	1	−2	2	+	1	1	[2 2 2 2]	1	0
−1	2	+	0	2	[2]	2	−2	2	−	1	0	[2 2 2 2]	2	0
−1	2	−	0	1	[2]	2	−2	2	+	0	2	[2 2]	2	0
−1	3	−	1	0	[3 3]	4	−2	2	−	0	0	[2 2]	4	0
−1	4	+	1	1	[2 4]	2	−2	2	−	0	1	[2 2]	2	0
−1	6	+	1	1	[2 3]	2	−2	2	+	−1	1	[]	4	1
							−2	2	+	−1	3	[]	4	1
							−2	2	−	−1	0	[]	7	1
							−2	2	−	−1	1	[]	4	1
							−2	2	−	−1	2	[]	4	1
							−2	3	−	0	0	[3]	6	0
							−2	4	−	1	0	[2 2 2]	2	2
							−2	4	+	1	1	[4 4]	4	0
							−2	4	−	1	0	[4 4]	8	0
							−2	4	+	0	2	[2]	2	0
							−2	4	−	0	0	[2]	8	0
							−2	4	−	0	1	[2]	2	0
							−2	6	+	1	1	[2 6]	2	0
							−2	6	−	1	0	[2 6]	4	0
							−2	6	+	1	1	[3 3]	4	4
							−2	6	−	1	0	[3 3]	4	4
							−2	8	−	1	0	[2 4]	4	4
							−2	12	−	1	0	[2 3]	4	4

of genus up to 13. The analytical results obtained in this paper coincide with numerical results obtained by generating all maps on surfaces of the corresponding types.

References

1. Bender E. and Canfield E., *The asymptotic number of rooted maps on a surface*, J. Combin. Theory Ser. A **43** (1986), 244–257.

2. Bender E., Canfield E. and Robinson R., *The enumeration of maps on the torus and the projective plane*, Canad. Math. Bull. **31** (1988), 257–271.

3. Breda d'Azevedo A., Mednykh A. and Nedela R., *Enumeration of maps regardless of genus: Geometric approach*, Discrete Math. **310** (2010), 1184–1203.

4. Bujalance E., *Cyclic groups of automorphisms of compact non-orientable Klein surfaces without boundary*, Pacific J. Math **109** (1983), 279–289.

5. Bujalance E., Cirre F., Gamboa M. and Gromadzki G., *Symmetries of Compact Riemann Surfaces*, Lecture Notes in Math., Springer, 2010.

6. Chalambides Ch., *Enumerative Combinatorics*, Chapman and Hall, 2002.

7. Etayo J., *Non-orientable automorphisms of Riemann Surfaces*, Arch. Math. (Basel) **45** (1985), 374–384.

8. Gao Z., *The number of rooted triangular maps on a surface*, J. Combin. Theory Ser. B **52** (1991), 236–249.

9. Giorgetti A. and Walsh T., *Efficient enumeration of rooted maps of a given orientable genus by number of faces and vertices*, Ars Math. Contemp. **7** (2014), 263–280.

10. Giorgetti A. and Walsh T., *Constructing large tables of numbers of maps by orientable genus*, `arXiv:1405.0615`.

11. Gross J. L., Yellen J. and Zhang P., *Handbook of Graph Theory*, Chapman and Hall/CRC, 2013.

12. Hall W., *Automorphisms and Coverings of Klein Surfaces*, PhD thesis, University of Southampton, Maths, 1977.

13. Jones G., *Enumeration of homomorphisms and surface-coverings*, Q. J. Math. **46** (1995), 485–507.

14. Krasko E., *Counting unlabelled chord diagrams of maximal genus*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **464** (2017), 77–87, (in Russian).

15. Krasko E. and Omelchenko A., *Enumeration of 4-regular one-face maps*, European J. Combin. **62** (2017), 67–177.

16. Krasko E. and Omelchenko A., *Enumeration of regular maps on surfaces of a given genus*, J. Math. Sci. (N.Y.) **232** (2018), 44–60.

17. Krasko E. and Omelchenko A., *Enumeration of r-regular maps on the torus. Part I: Rooted maps on the torus, the projective plane and the Klein bottle. Sensed maps on the torus*, Discrete Math. **342** (2019), 584–589.

18. Krasko E. and Omelchenko A., *Enumeration of r-regular maps on the torus. Part II: Unsensed maps*, Discrete Math. **342** (2019), 600–614.

19. Liskovets V., *Enumeration of nonisomorphic planar maps*, Selecta Math. Sovietica **4** (1985), 303–323.

20. Liskovets V., *A reductive technique for enumerating non-isomorphic planar maps*, Discrete Math. **156** (1996), 197–217.

21. Mednykh A. and Nedela R., *Enumeration of unrooted maps of a given genus*, J. Combin. Theory Ser. B **96** (2006), 709–729.

22. Mednykh A. and Nedela R., *Enumeration of unrooted hypermaps of a given genus*, Discrete Math. **310** (2010), 518–526.

23. Tutte, W., *A census of planar triangulations*, Canad. J. Math. **14** (1962), 21–38.

24. Tutte, W., *A census of slicings*, Canad. J. Math. **14** (1962), 708–722.

25. Tutte, W., *A census of planar maps*, Canad. J. Math. **15** (1963), 249–271.

26. Walsh, T., *Space-efficient generation of nonisomorphic maps and hypermaps*, J. Integer Seq. **18** (2015).

27. Walsh T. and Lehman A., *Counting rooted maps by genus, I*, J. Combin. Theory Ser. B **13** (1972), 192–218.

E. Krasko, Computer Science Department, Saint Petersburg School of Physics,
Mathematics and Computer Science, National Research University Higher School of Economics,
Saint Petersburg, Russia,
e-mail: `krasko.evgeniy@gmail.com`

A. Omelchenko, Computer Science Department, Saint Petersburg School of Physics,
Mathematics and Computer Science, National Research University Higher School of Economics,
Saint Petersburg, Russia,
e-mail: `avo.travel@gmail.com`