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ON A FRANKL-WILSON THEOREM AND ITS GEOMETRIC COROLLARIES

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Abstract. We find a new analogue of the Frankl-Wilson theorem on the independence number of distance graphs of some special type. We apply this new result to two combinatorial geometry problems.

First, we improve a previously known value c such that $\chi(\mathbb{R}^n; S_2) \geq (c + o(1))^n$, where $\chi(\mathbb{R}^n; S_2)$ is the minimum number of colors needed to color all points of $\mathbb{R}^n$ so that there is no monochromatic set of vertices of a unit equilateral triangle S_2.

Second, given $m \geq 3$ we improve the value ξ_m such that for any $n \in \mathbb{N}$ there is a distance graph in $\mathbb{R}^n$ with the girth greater than m and the chromatic number at least $(\xi_m + o(1))^n$.

1. Main theorem

Given natural numbers $n \geq k \geq t$ let us define a distance graph $G(n,k,t) = (V(n,k), E(n,k,t))$ as follows.

$$V(n,k) = \left\{ v = (v_1, \ldots, v_n) \in \mathbb{R}^n : \forall\, i\ v_i \in \{0,1\} \text{ and } \sum_{i=1}^{n} v_i = k \right\},$$

$$E(n,k,t) = \left\{ \{v,w\} : v, w \in V(n,k) \text{ and } \sum_{i=1}^{n} v_i w_i = t \right\}.$$

One can easily see that

$$|V(n,k)| = \binom{n}{k}, \qquad |E(n,k,t)| = \frac{1}{2}\binom{n}{k}\binom{k}{t}\binom{n-k}{k-t}.$$

Graphs $G(n,k,t)$ are among of the most important graphs for modern extremal combinatorics for many reasons. Firstly, they are used in coding theory (see [**18**]). Secondly, these graphs provide asymptotically the best counterexamples to Borsuk's conjecture (see [**19**]). And last but not least, graphs $G(n,k,t)$ are applicable for Nelson's problem and related problems on chromatic numbers of spaces (see [**8, 9, 30, 31, 32**] and §2 of this paper).

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One of the most important cases for applications is the case $k = k(n) \sim \kappa n$, $t = t(n) \sim \tau n$ as $n \to \infty$, where $0 < \tau < \kappa \leq \frac{1}{2}$ are fixed parameters.

One can use Stirling's formula to see that

$$\binom{an + o(n)}{bn + o(n)} = \left(c_a^b + o(1)\right)^n \text{ as } n \to \infty, \qquad \text{where } c_a^b = \frac{a^a}{b^b (a-b)^{(a-b)}}.$$

Hence, in our special case one has

$$|V(n,k)| = \left(c_1^\kappa + o(1)\right)^n, \qquad |E(n,k,t)| = \left(c_1^\kappa c_\kappa^\tau c_{1-\kappa}^{\kappa-\tau} + o(1)\right)^n.$$

We combine the results that estimate the independence number of $G(n,k,t)$ in our case into the following theorem.

Theorem 1. *Let $k = k(n) \sim \kappa n$, $t = t(n) \sim \tau n$ as $n \to \infty$, where $0 < \tau < \kappa \leq \frac{1}{2}$ are fixed. Let the value $k - t$ be a prime number for every n. Then*

$$\alpha\left(G(n,k,t)\right) \leq \begin{cases} \left(c_1^{\kappa-\tau} + o(1)\right)^n & \text{if } 2\tau \leq \kappa, \\ \left(\dfrac{c_1^\kappa c_1^{\kappa-\tau}}{c_1^{2\kappa-2\tau}} + o(1)\right)^n & \text{if } 2\tau > \kappa. \end{cases}$$

The first item of Theorem 1 is due to Frankl and Wilson (see [**5**]). It was proven in [**2**], [**3**] that this bound is best possible. The second item is due to [**14**] and [**20**]. It is not known whether this bound can be further improved or not.

It follows from Theorem 1 that any sufficiently large subset $W \subset V(n,k)$ contains at least one edge inside. Our new Theorem 2 improves this result and states that one can find almost all edges inside any sufficiently large subset $W \subset V(n,k)$.

Theorem 2. *Let $k = k(n) \sim \kappa n$, $t = t(n) \sim \tau n$ as $n \to \infty$, where $0 < \tau < \kappa \leq \frac{1}{2}$ are fixed. Let the value $k - t$ be a prime number for every n. Then there is a function $\delta = \delta(\varepsilon, \kappa, \tau) > 0$ such that two following statements are valid. First, given $\varepsilon > 0$ any subset $W \subset V(n,k)$ such that $|W| \geq (1-\delta)^n |V(n,k)|$ contains at least $(1-\varepsilon)^n |E(n,k,t)|$ edges inside for every sufficiently large n. Second,*

$$\delta(\varepsilon, \kappa, \tau) = \frac{1 + o(1)}{4(\kappa - \tau)} \frac{\varepsilon^2}{\ln^2 \varepsilon}$$

as $\varepsilon \to 0$.

We believe that our main Theorem 2 is of independent interest. Besides, in §2 we give two examples of applying this theorem to combinatorial geometry problems.

The first non explicit result similar to Theorem 2 was obtained by Frankl and Rödl in [**6**]. Similar, but much weaker explicit results were obtained in [**6, 23, 24, 33**]. Also see related works in [**1, 10, 16, 15, 21, 25, 27, 28**].

2. Geometrical applications

In 1950 Nelson asked what is the minimum number of colors $\chi\left(\mathbb{R}^2\right)$ needed to color all points of $\mathbb{R}^2$ so that no two points at distance one receive the same color.

Despite the fact that this problem does not look very hard the answer is still unknown. Nowadays we only know (see [**7**]) that

$$5 \leq \chi\left(\mathbb{R}^2\right) \leq 7.$$

A lot of generalizations of this original problem have been considered over the last seventy years. One of them is as follows. Let $S \subset \mathbb{R}^d$ be a finite set. Define the chromatic number $\chi\left(\mathbb{R}^n; S\right)$ as the minimum number of colors needed to color all points of $\mathbb{R}^n$ $(n \geq d)$ so that no congruent copy of S in $\mathbb{R}^n$ is monochromatic. The set S is called exponentially Ramsey if there is a constant $c = c(S) > 1$ such that $\chi\left(\mathbb{R}^n; S\right) > (c + o(1))^n$ as $n \to \infty$. A criterion that could distinguish all exponentially Ramsey sets is still unknown (see [**13**]). However, many examples of exponentially Ramsey sets are known.

For example, Frankl and Wilson proved in [**5**] that the set S_1 that consists of two points at unit distance is exponentially Ramsey. It was proven in [**12**] and [**22**] that

$$(1.239\ldots + o(1))^n < \chi\left(\mathbb{R}^n; S_1\right) < (3 + o(1))^n$$

as $n \to \infty$. Frankl and Rödl proved in [**6**] that the set S_2 of vertices of a unit equilateral triangle is exponentially Ramsey. It was proven in [**17**] and [**26**] that

$$(1.00085\ldots + o(1))^n < \chi\left(\mathbb{R}^n; S_2\right) < (2.732\ldots + o(1))^n.$$

Our new Theorem 3 improves the lower bound from the last inequality. This theorem is our first geometrical application of Theorem 2 in this section.

Theorem 3. *One has*

$$\chi\left(\mathbb{R}^n; S_2\right) > (1.014\ldots + o(1))^n, \qquad \textit{as } n \to \infty.$$

The second geometrical application of Theorem 2 is as follows. Following the paper [**4**] we define $\xi_m(\mathbb{R}^n)$ as the maximum among the chromatic numbers of distance graphs in $\mathbb{R}^n$ with the girth greater than m, where $m \geq 3$. Kupavskii proved in [**11**] that there is $c_m > 1$ such that

$$\xi_m(\mathbb{R}^n) > (c_m + o(1))^n$$

as $n \to \infty$. Let ξ_m be the supremum of constants c_m such that the previous inequality holds. Let us define this value more precisely.

Given $m \geq 3$ we define $\mathcal{G}(n,m)$ as the family of all distance graphs in $\mathbb{R}^n$ that do not contain cycles of length less than or equal to m. Then

$$\xi_m = \liminf_{n\to\infty} \max_{G \in \mathcal{G}(n,m)} \chi(G)^{1/n}.$$

It was proven in [**26**] and [**29**] that

$$\xi_m > 1 + \frac{0.0133\ldots + o(1)}{m^2 \ln^2 m},$$

as $m \to \infty$. Our Theorem 2 can be used to replace the constant 0.0133 from the numerator by a much larger value.

Theorem 4. *One has*

$$\xi_m > 1 + \frac{0.632\ldots + o(1)}{m^2 \ln^2 m}, \qquad \text{as } m \to \infty.$$

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