Acta Math. Univ. Comenianae
Vol. LXXXVIII, 3 (2019), pp. 533–539

CYCLES OF LENGTH THREE AND FOUR IN TOURNAMENTS

T. F. N. CHAN, A. GRZESIK, D. KRÁL' AND J. A. NOEL

ABSTRACT. Linial and Morgenstern conjectured that, among all n-vertex tournaments with $d\binom{n}{3}$ cycles of length three, the number of cycles of length four is asymptotically minimized by a random blow-up of a transitive tournament with all but one part of equal size and one smaller part. We prove the conjecture for $d \geq 1/36$ by analyzing the possible spectrum of adjacency matrices of tournaments. We also demonstrate that the family of extremal examples is broader than expected and give its full description for $d \geq 1/16$.

1. INTRODUCTION

One of the oldest theorems in extremal graph theory is Mantel's theorem [**14**], which asserts that every n-vertex graph with more than $n^2/4$ edges contains a triangle. The Erdős–Rademacher Problem, which can be traced back to the work of Rademacher in the 1940's and the later work of Erdős [**4**], asks for the minimum possible number of triangles in a graph with a given number of vertices and edges. It was conjectured that this minimum is asymptotically attained by a complete multipartite graph with all but one part of equal size and one smaller part. This conjecture attracted substantial attention for several decades, see e.g. [**1, 5, 8, 12**], until its solution by Razborov [**17**] using his newly developed flag algebra method. Pikhurko and Razborov [**16**] described the asymptotic structure of all extremal graphs and an exact description was obtained in [**10**]. The more general problem of determining the minimum asymptotic density of k-cliques in graphs with given edge-density (the Erdős–Rademacher Problem corresponds to the case $k = 3$) has also been solved by Nikiforov [**15**] (the case $k = 4$) and by Reiher [**18**] in full generality.

Received May 31, 2019.
2010 *Mathematics Subject Classification.* Primary 05C20; Secondary 05C35.
This publication is an extended abstract of a contribution accepted for presentation at the 10th European Conference on Combinatorics, Graph Theory and Applications 2019. This work was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 648509) and by the Engineering and Physical Sciences Research Council Standard Grant number EP/M025365/1. This publication reflects only its authors' view; the European Research Council Executive Agency is not responsible for any use that may be made of the information it contains. The third author was also supported by the MUNI Award in Science and Humanities of the Grant Agency of Masaryk university, and the fourth author was also supported by the Leverhulme Trust Early Career Fellowship ECF-2018-534.

We investigate a similar problem for tournaments posed by Linial and Morgenstern [**9**], who asked for the minimum density of 4-cycles in a large tournament with fixed density of 3-cycles. They conjectured that the tournament asymptotically minimizing this density is a blow-up of a transitive tournament with all but one part of equal size and one smaller part in which the arcs within each part are oriented randomly (they call this construction a *random blow-up*), i.e., the structure of the conjectured extremal examples is akin to those of the Erdős–Rademacher problem.

We confirm this conjecture in the case where the proposed extremal examples have two or three parts and provide a full description of extremal tournaments in the two part case. In contrast to many of the recent proofs in this area that use the flag algebra method, our approach is based on the analysis of the spectrum of adjacency matrices of tournaments.

2. Statement of the problem

We now state the problem that we study formally. The *density* of the directed cycle C_ℓ of length ℓ in a tournament T, denoted by $t(C_\ell, T)$, is the probability that a random mapping from $V(C_\ell)$ to $V(T)$ is a homomorphism (i.e. arcs of C_ℓ map to arcs of T). Note that, for fixed ℓ, a tournament T on n vertices contains $t(C_\ell, T)n^\ell/\ell + O(n^{\ell-1})$ cycles of length ℓ.

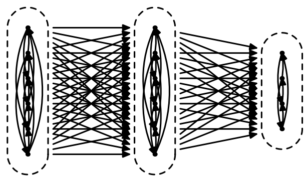

Figure 1. An illustration of the random blow up construction for $z = 3/8$.

Our focus is on bounding the minimum possible value of $t(C_4, T)$ asymptotically as a function of $t(C_3, T)$. To motivate the definition of a function g later, we describe the family of conjectured tight examples from [**9**] for this problem. Fix $z \in [0, 1]$ and $n \in \mathbb{N}$. We define an n-vertex tournament T as follows (see Figure 1 for an illustration). If $z = 0$, then let T be a transitive tournament. Otherwise, the vertices of T are split into $\lfloor z^{-1} \rfloor + 1$ disjoint parts $V_1, \ldots, V_{\lfloor z^{-1} \rfloor + 1}$ such that $\lfloor z^{-1} \rfloor$ parts contain exactly $\lfloor zn \rfloor$ vertices and the remaining part contains the rest of the vertices (note that if z^{-1} and zn are integers, then the last part is empty). If two vertices v and v' belong to distinct parts V_i and V_j with $i < j$, then the tournament T contains an edge from v to v'. If v and v' belong to the same part, then the edge between them is oriented from v to v' with probability $1/2$, i.e., each part itself induces a random tournament. It is easy to see that $t(C_3, T) = t(C_4, T) = 0$

if $z = 0$ and, if $z \in (0, 1]$, then, with high probability, it holds that

$$t(C_3, T) = \frac{1}{8}\left(\lfloor z^{-1}\rfloor z^3 + \left(1 - \lfloor z^{-1}\rfloor z\right)^3\right) + o(1) \qquad \text{and}$$
$$t(C_4, T) = \frac{1}{16}\left(\lfloor z^{-1}\rfloor z^4 + \left(1 - \lfloor z^{-1}\rfloor z\right)^4\right) + o(1).$$

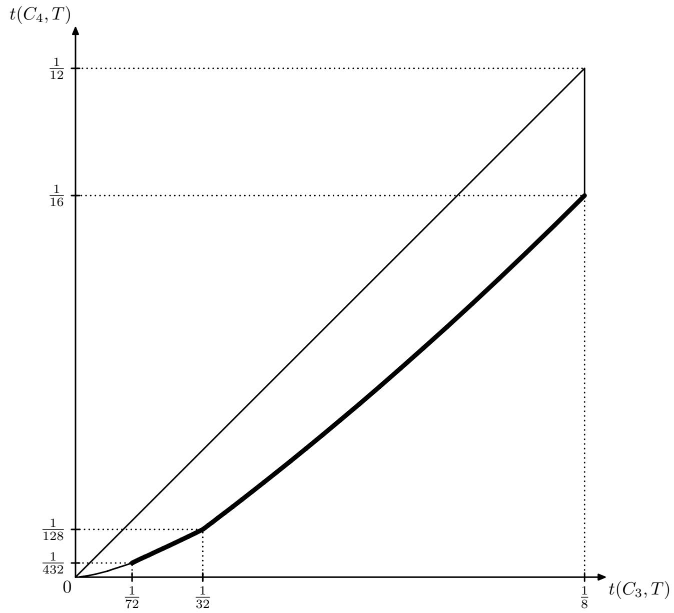

Figure 2. The conjectured region of asymptotically feasible densities of C_3 and C_4 in tournaments. The lower bound for $t(C_3, T) \in \{1/8, 1/32\}$ and the upper bound were proved in [**9**]. The rest of the lower bound is conjectured except for the part depicted in bold, which we prove here.

Linial and Morgenstern [**9**] conjectured that the above construction is asymptotically optimal. To state their conjecture, we define a function $g\colon [0, 1/8] \to [0, 1]$ as follows: $g(0) = 0$ and

$$g\left(\frac{1}{8}\left(\lfloor z^{-1}\rfloor z^3 + \left(1 - \lfloor z^{-1}\rfloor z\right)^3\right)\right) = \frac{1}{16}\left(\lfloor z^{-1}\rfloor z^4 + \left(1 - \lfloor z^{-1}\rfloor z\right)^4\right)$$

for $z \in (0, 1]$.

Note that $t(C_3, T) \leq 1/8$ for every tournament T.

Conjecture 1 (Linial and Morgenstern [**9**, Conjecture 2.2])**.** It holds that

$$t\left(C_4, T\right) \geq g\left(t(C_3, T)\right) + o(1)$$

for every tournament T.

The conjecture is currently only known to hold for tournaments with 3-cycle density asymptotically equal to 1/8 or 1/32 [**9**]. Our results confirm the conjecture for all 3-cycle densities in the range $[1/72, 1/8]$, i.e., in the regimes of two and three parts. We refer to Figure 2 for the visualization of our results and the conjectured feasible region of 3-cycle and 4-cycle densities.

Conjecture 1 appears to be resistant to the flag algebra method (our proof uses an analysis of spectra of adjacency matrices of tournaments). We believe that the difficulty in applying the flag algebra method is rooted in the fact that random blow-ups of transitive tournaments are not the only extremal examples for Conjecture 1. In particular, a rather complicated family of extremal examples T is described as follows. Denote the vertices of T by $v_1, \dots, v_n$ and associate v_i with a real number $p_i \in [0, 1/2]$, $i = 1, \dots, n$. Then, direct the edge v_iv_j from v_i to v_j with probability $1/2 + p_i - p_j$. Note that, if all the values of p_i are either 0 or $1/2$, then this construction is nothing more than a random blow-up of a 2-vertex tournament, i.e., it is identical to the examples of [**9**] for 3-cycle density in $[1/32, 1/8]$. For large n, this tournament satisfies $t(C_4, T) = g(t(C_3, T)) + o(1)$ with high probability (this follows from Theorem 8). In particular, all tournaments obtained in this way are extremal with respect to Conjecture 1 in the regime of two parts. In Corollary 9, we prove that these are asymptotically the only extremal constructions in the regime of two parts.

3. Tournament matrices and tournament limits

In this section, we introduce the notation that we use, including the notions of tournament matrices and tournament limits. If A is a matrix, then we write A^T for its transpose. The *trace* of a square matrix A is the sum of the entries in its diagonal and is denoted by $\operatorname{Tr} A$. We use $\mathbb{J}_n$ to denote the square matrix of order n such that each entry of $\mathbb{J}_n$ is equal to one; if n is clear from the context, we will omit the subscript. We say that a square matrix A of order n is a *tournament matrix* if A is non-negative and $A + A^T = \mathbb{J}$; in particular, if A is a tournament matrix, then each diagonal entry of A is equal to $1/2$. Every n-vertex tournament T can be associated with a tournament matrix A of order n, which we refer to as the *adjacency matrix* of T, in the following way. Each diagonal entry A is equal to $1/2$ and, for $i \neq j$, the entry of A in the i-th row and the j-th column (denoted $A_{i,j}$) is equal to 1 if T contains an edge oriented from the i-th vertex to the j-th vertex, and it is equal to 0 otherwise. The following proposition readily follows.

Proposition 2. *Let T be a tournament on n vertices, A be the adjacency matrix of T and $\ell \geq 3$. The number of homomorphisms of C_ℓ to T is* $\operatorname{Tr} A^\ell + O(n^{\ell-1})$.

Recall that the trace of a matrix is equal to the sum of its eigenvalues and that the eigenvalues of the ℓ-th power of a matrix are the ℓ-th powers of its eigenvalues. In view of Proposition 2, for $\ell \geq 1$, we define $\sigma_\ell(A)$ for a square matrix A of order n to be

$$\sigma_\ell(A) = \frac{1}{n^\ell} \sum_{i=1}^{n} \lambda_i^\ell = \frac{1}{n^\ell} \operatorname{Tr} A^\ell,$$

where $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are the eigenvalues of A. Note that the normalization of the sum is chosen in such a way that $\sigma_1(A) = 1/2$ for every tournament matrix A.

It is not hard to see that Conjecture 1 is equivalent to the following.

Conjecture 3. If A is a tournament matrix, then $\sigma_4(A) \geq g(\sigma_3(A))$.

To describe the asymptotically optimal tournaments in the regime of two parts, we use the language of the theory of combinatorial limits. Below, we define basic concepts concerning tournament limits. These are analogous to those concerning graph limits, which can be found, e.g., in the monograph on graph limits by Lovász [**11**]. In fact, most statements on graph limits readily translate to the setting of tournaments with essentially the same proofs.

A *tournament limit* is a measurable function $W\colon [0,1]^2 \to [0,1]$ such that $W(x,y) + W(y,x) = 1$ for all $(x,y) \in [0,1]^2$. One can define the density of the cycle C_ℓ in W as follows:

$$t(C_\ell, W) = \int_{x_1,\ldots,x_\ell \in [0,1]} W(x_1,x_2)W(x_2,x_3)\cdots W(x_{\ell-1},x_\ell)W(x_\ell,x_1)\,\mathrm{d}x_1\cdots x_\ell\,.$$

Note that any n-dimensional tournament matrix A can be represented by a tournament limit W_A by dividing $[0,1]$ into sets $I_1,\ldots,I_n$ of measure $1/n$ and setting W equal to $A_{i,j}$ on the set $I_i \times I_j$. It is easily observed that $t(C_\ell, W_A)$ is precisely $\sigma_\ell(A)$. The following proposition links densities of cycles in tournament limits and in tournaments.

Proposition 4. *The following two statements are equivalent for every sequence $(s_\ell)_{\ell\geq 3}$ of non-negative reals:*

- *There exists a tournament limit W such that $t(C_\ell, W) = s_\ell$ for every $\ell \geq 3$.*
- *There exists a sequence $(T_i)_{i\in\mathbb{N}}$ of tournaments with increasing orders such that*
$$\lim_{i\to\infty} t(C_\ell, T_i) = s_\ell \qquad \textit{for every } \ell \geq 3.$$

Hence, another equivalent formulation of Conjecture 1 is the following.

Conjecture 5. It holds $t(C_4, W) \geq g(t(C_3, W))$ for every tournament limit W.

4. Main results

The proof of Conjecture 1 in the regimes of two and three parts is based on the following theorem on spectra of tournament matrices.

Theorem 6. *Let A be a tournament matrix of order n. If $\sigma_3(A) \geq 1/72$, then $\sigma_4(A) \geq g(\sigma_3(A))$.*

An immediate corollary of Theorem 6 in the setting of tournament limits is the following.

Corollary 7. *Let W be a tournament limit. If $t(C_3, W) \geq 1/72$, then it holds that $t(C_4, W) \geq g(t(C_3, W))$.*

In the regime of two parts, we can characterize tournament matrices where the equality holds.

Theorem 8. *Let A be a tournament matrix of order n. If $\sigma_3(A) \geq 1/32$, then $\sigma_4(A) \geq g(\sigma_3(A))$ with equality if and only if there exists a vector $z \in \mathbb{R}^n$ such that $A_{i,j} = 1/2 + z_i - z_j$ for $i, j \in [n]$.*

Interpreting Theorem 8 in the language of tournament limits, we obtain the following corollary.

Corollary 9. *Let W be a tournament limit. If $t(C_3, W) \geq 1/32$, then it holds that $t(C_4, W) \geq g(t(C_3, W))$ and the equality holds if and only if there exists a measurable function $f\colon [0,1] \to [0,1/2]$ such that $W(x,y) = 1/2 + f(x) - f(y)$ for almost every $(x,y) \in [0,1]^2$.*

We finish with a high level overview of the proofs of Theorems 6 and 8. To prove Theorem 6, we consider the following optimization problem involving complex numbers $z_1, \dots, z_n$. The goal is to minimize $z_1^4 + \cdots + z_n^4$ subject to the following constraints: each z_i has a non-negative real part, $z_1 + \cdots + z_n = 1/2$, $z_1^3 + \cdots + z_n^3 = \sigma_3(A)$, if z_i has a positive complex part, then $z_{i+1} = \overline{z_i}$, and if z_i has a negative complex part, then $z_{i-1} = \overline{z_i}$. The problem is set up in a way that if A is a tournament matrix of order n with eigenvalues $\lambda_1, \dots, \lambda_n$, then $\lambda_1/n, \dots, \lambda_n/n$ form a feasible solution. The analysis of optimal solutions of this problem yields the proof of Theorem 6.

To prove Theorem 8, we consider the skew-symmetric matrix B defined as $\mathbb{J} - 2A$. Informally, the matrix B reflects how much the tournament differs from the quasirandom tournament. The traces of the third and fourth powers of A are related to B as follows:

$$8 \operatorname{Tr} A^3 = \operatorname{Tr}(\mathbb{J} - B)^3 = \operatorname{Tr} \mathbb{J}^3 + 3 \operatorname{Tr} \mathbb{J} B^2 \text{ and}$$

$$16 \operatorname{Tr} A^4 = \operatorname{Tr} \mathbb{J}^4 + 4 \operatorname{Tr} \mathbb{J}^2 B^2 + \operatorname{Tr} B^4 = -\frac{1}{3} \operatorname{Tr} \mathbb{J}^4 + \frac{32}{3} \operatorname{Tr} A^3 + \operatorname{Tr} B^4 .$$

Hence, if the order of A and the value of $\operatorname{Tr} A^3$ are fixed, the goal is to minimize $\operatorname{Tr} B^4$, i.e., the sum of the fourth powers of the eigenvalues of B. The analysis of this optimization problem involving the spectrum of B is performed using the block diagonal form of skew-symmetric matrices.

References

1. Bollobás B., *Relations between sets of complete subgraphs*, in: Proceedings of the 5th British Combinatorial Conference, 1976, 79–84.
2. Brauer A. and Gentry I. C., *On the characteristic roots of tournament matrices*, Bull. Amer. Math. Soc. **74** (1968), 1133–1135.
3. Chung F. R. K. and Graham R. L., *Quasi-random tournaments*, J. Graph Theory **15** (1991), 173–198.
4. Erdős P., *Some theorems on graphs*, Riveon Lematematika **9** (1955), 13–17.
5. Fisher D. C., *Lower bounds on the number of triangles in a graph*, J. Graph Theory **13** (1989), 505–512.
6. Frieze A. and Kannan R., *Quick approximation to matrices and applications*, Combinatorica **19** (1999), 175–200.
7. Gantmacher F. R., *The Theory of Matrices*, translated from the Russian by K. A. Hirsch, Reprint of the 1959 translation, vol. 1, 1998.

8. Goodman A. W., *On sets of acquaintances and strangers at any party*, Amer. Math. Monthly **66** (1959), 778–783.
9. Linial N. and Morgenstern A., *On the number of 4-cycles in a tournament*, J. Graph Theory **83** (2016), 266–276.
10. Liu H., Pikhurko O. and Staden K., *The exact minimum number of triangles in graphs of given order and size*, arXiv:1712.00633.
11. Lovász L., *Large Networks and Graph Limits*, Colloquium Publications vol. 60, AMS, 2012.
12. Lovász L. and Simonovits M., *On the number of complete subgraphs of a graph. II*, in: Studies in Pure Mathematics, 1983, 459–495.
13. Lovász L. and Szegedy B., *Limits of dense graph sequences*, J. Combin. Theory Ser. B **96** (2006), 933–957.
14. Mantel W., *Problem 28*, Wiskundige Opgaven **10** (1907), 60–61.
15. Nikiforov V. S., *The number of cliques in graphs of given order and size*, Trans. Amer. Math. Soc. **363** (2011), 1599–1618.
16. Pikhurko O. and Razborov A., *Asymptotic structure of graphs with the minimum number of triangles*, Combin. Probab. Comput. **26** (2017), 138–160.
17. Razborov A., *On the minimal density of triangles in graphs*, Combin. Probab. Comput. **17** (2008), 603–618.
18. Reiher C., *The clique density theorem*, Ann. Math. **184** (2016), 683–707.

T. F. N. Chan, School of Mathematical Sciences, Monash University, Melbourne, Australia
Mathematics Institute and DIMAP, University of Warwick, Coventry, UK,
e-mail: timothy.chan@monash.edu

A. Grzesik, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland,
e-mail: Andrzej.Grzesik@uj.edu.pl

D. Kráľ, Faculty of Informatics, Masaryk University, Brno, Czech Republic
Mathematics Institute, DIMAP and Department of Computer Science, University of Warwick, Coventry UK,
e-mail: dkral@fi.muni.cz

J. A. Noel, Mathematics Institute and DIMAP, University of Warwick, Coventry, UK,
e-mail: J.Noel@warwick.ac.uk