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HOMOMORPHISM-HOMOGENEITY CLASSES OF COUNTABLE L-COLORED GRAPHS

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ABSTRACT. The notion of homomorphism-homogeneity, introduced by Cameron and Nešetřil, originated as a variation on ultrahomogeneity. By fixing the type of finite homomorphism and global extension, several homogeneity classes, called morphism extension classes, can be defined. These classes are studied for various languages and axiom sets. Hartman, Hubička and Mašulović showed for finite undirected L-colored graphs without loops, where colors for vertices and edges are chosen from a partially ordered set L, that when L is a linear order, the classes HH and MH of L-colored graphs coincide, contributing thus to a question of Cameron and Nešetřil. They also showed that the same is true for vertex-uniform finite L-colored graphs when L is a diamond. In this work, we extend their results to countably infinite L-colored graphs, proving that the classes MH and HH coincide if and only if L is a linear order.

Some interesting graph properties are represented by various versions of structural symmetry. An example is vertex transitivity, the condition that for every pair of vertices there exists an automorphism of the graph mapping one to the other. These symmetry conditions are interesting on their own and have applications in other fields where definable graphs may appear. Ultrahomogeneity is one of the strongest notions of symmetry. A graph is *ultrahomogeneous* if for any isomorphism f between finite induced subgraphs there is an automorphism of the ambient graph that extends f. A countable ultrahomogeneous relational structure is ω-categorical and eliminates quantifiers, so ultrahomogeneous structures are interesting from the point of view of model theory and group theory.

The fundamental theorem of ultrahomogeneity, which states the equivalence between ultrahomogeneity in countable relational structures and some properties of the set of induced finite substructures, was proven by Fraïssé [**8**]. For any given class of relational structures, one naively expects only a few of them to satisfy such a strong symmetry condition, and this is indeed the case for most of the classes that have been studied so far (but families with a maximal number of ultrahomogeneous members exist, for example directed graphs [**6**]). For example, other than the obvious finite cases of complete graphs, complete bipartite graphs,

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and their complements, only C_5 and the line graph of $K_{3,3}$ are ultrahomogenous [**9**]. In the case of countably infinite undirected graphs, there are only countably many ultrahomogeneous graphs, and they fall neatly in a few families, as shown by the celebrated Lachlan-Woodrow theorem [**13**]. Ultrahomogeneous structures have been heavily studied in the past decade from many points of view, such as the study of CSPs with ultrahomogeneous templates by Bodirsky [**3**], relational complexity (introducing new predicates to homogenize a structure while minimizing arities of the new relations, [**11**]) and the related connections to the Ramsey property by Hubička and Nešetřil [**12**].

If we take the definition of ultrahomogeneity and shift it from the category of L-structures with isomorphisms to the category of L-structures with homomorphisms, we obtain the definition of homomorphism-homogeneity, introduced by Cameron and Nešetřil [**4**]. More formally, an L-structure is homomorphism-homogeneous if any homomorphism between finite induced substructures is the restriction of an endomorphism. We can refine this definition by restricting the kind of local homomorphism (we could add no more hypotheses or require it to be injective or an isomorphism) and endomorphism. Lockett and Truss [**14**] did exactly this to define several morphism-extension classes. Following the notation from [**4**], they use a pair of characters XY to denote the morphism-extension class in which any local X-morphism is restriction of a a global Y-morphism. Here $\mathrm{X} \in \{\mathrm{H}, \mathrm{M}, \mathrm{I}\}$ stands for *homo, mono* or *iso* and $\mathrm{Y} \in \{\mathrm{H}, \mathrm{A}, \mathrm{I}, \mathrm{B}, \mathrm{E}, \mathrm{M}\}$ stands for *homo, auto, iso, bi, epi* or *mono.* Thus, for example, the notion of homomorphism-homogeneity above is what we will call HH-homogeneity, and ultrahomogeneity is IA-homogeneity.

HH-homogeneity has not been studied as intensely as ultrahomogeneity. While we have a classification of finite HH-homogeneous graphs (not very interesting since only complete and empty finite graphs are HH-homogeneous [**4**]), we do not have an analogue of the Lachlan-Woodrow theorem for HH. In fact, only two families of connected HH-homogeneous graphs appear in the literature, namely those graphs that contain the Rado graph as a spanning subgraph [**4**] and the Rusinov-Schweitzer examples [**18**]. Recently the authors have produced new families of connected countably infinite HH-homogeneous graphs that do not contain the Rado graph as a spanning subgraph, one of them parametrized by two natural numbers [**2**]. Homomorphism-homogeneity has also been studied in different contexts, e.g. HH-homogeneity for infinite-domain CSP [**16**] or equivalence of two types of convergence for a graph sequence that converge elementarily to an IH-homogeneous graph [**15**].

Since any monomorphism is a homomorphism we can see that HH is always a subclass of MH, and we can partially order morphism-extension classes using $\subseteq$, as presented in greater extent in [**14**]. Already in the original paper by Cameron and and Nešetřil [**4**] the question was asked: Do MH and HH coincide for countable undirected graphs? Rusinov and Schweitzer answered the question in the affirmative in [**18**]. We explore this question for countably infinite L-colored graphs. Full versions of the arguments can be found at [**1**].

We will consider relational structures with finitely many predicates that are partially ordered, so the language and the partial order will be used interchangeably. If $\mathcal{L}$ is a finite partially ordered language with maximal arity 2, an $\mathcal{L}$-structure consists of a countably infinite set and mutually exclusive from $\mathcal{L}$. Moreover, the binary relations are irreflexive and symmetric. The role of the partial order on the language is to model multiple edges of different type on the same pair of vertices (or multiple unary predicates on a vertex), so we will modify our notion of homomorphism accordingly (see Definition 2 below).

We will not mention the language when it is clear from the context or when we use these properties as an adjective. Abbreviated statements like "G is MH" or "G is MH-homogeneous" mean that the structure G belongs to the class $\mathrm{MH}_{\mathcal{L}}$, where $\mathcal{L}$ is the appropriate language (which further below will be identified with one or two finite partial orders).

Definition 1. A P, Q-colored graph is a tuple (V, P, Q, χ, ξ), where V is a vertex set, P and Q are two disjoint finite partially ordered sets, $\chi\colon V \to P$ is an arbitrary function, and $\xi\colon V^2 \to Q$ is a symmetric function with $\xi(v, v) = \mathbb{0}$ for all $v \in V$. Our partial orders are always finite and have a minimum element $\mathbb{0}$ (corresponding to uncolored vertices and nonedges).

We will say that a P, Q-colored graph M is vertex-uniform if χ is constant.

Definition 2. A homomorphism between (G, P, Q, χ, ξ) and (H, P, Q, χ', ξ') is a function $f\colon G \to H$, such that for all $v \in G$, $\chi(v) \leq_P \chi'(f(v))$ and for all pairs $\{x, y\} \in G^2$, $\xi(x, y) \leq_Q \xi'(f(x), f(y))$.

Note that in Definition 2 we require the same "language," i.e., the same pair of partial orders, in both structures. For structures being MH we will use a symbol $\mathrm{MH}_{P,Q}$ in case of P, Q-colored graphs and (by convention) MH_Q in case of Q-colored graph, i.e., when vertices have no or same colors assigned.

For <u>finite</u> Q-colored graphs where Q is either a linear order or a diamond, i.e., set of incomparable elements extended by a minimal and a maximal elements, it was been shown in [**10**] that the classes MH_Q and HH_Q coincide. However, there are examples of finite Q-colored graph, i.e., without colors for vertices, which are HH and not MH. This led authors to ask the following question

Problem 3. Do MH and HH coincide for countably infinite Q-colored graphs with all vertices uncolored?

Depending on the answer to the above stated problem authors have asked the following general question.

Problem 4. Do MH and HH coincide for countably infinite P, Q-colored graphs?

We address both questions in our work, answering negatively for both problems. Moreover, we identify the subclass where the equality holds.

Theorem 5. *Let P and Q be finite partially ordered sets.* $\mathrm{MH}_{P,Q} = \mathrm{HH}_{P,Q}$ *if and only if Q is a linear order.*

Since the first paper on homorphism-homogeneity by Cameron and Nešetřil [**4**] there is an ongoing discussion on the relationship between various homomorphism-homogeneous classes. Some of the central relationships have been explained for countable undirected graphs [**18**] and others have been studied to a particular depth for more general structures [**14**]. This is of interest also due to recent studies of Fraïssé-type theorems for various classes of homomorphism-homogeneity [**17, 7**] resulting in countable structures in a given morphism-extension class. Considering how we modified the notion of homomorphism to model graphs with multiple types of edge allowed between two vertices, Theorem 5 tells us that the coincidence of MH and HH is a rare occurrence.

To show that equality of classes given in Theorem 5 is implied by a total order on Q we use a careful handling of specific transversals for a local homomorphism that is about to be extended. For the converse, i.e. assuming that Q is a linear order, we make use of Lemma 6 below. Let us remind that a partially ordered set is a *directed set* if every pair of its elements has an upper bound.

Lemma 6. *Let Q and P be finite partially ordered. If* $\mathrm{MH}_{P,Q} = \mathrm{HH}_{P,Q}$*, then Q is a directed set.*

The idea behind this lemma is that a finite partially ordered set that is not a directed set has several maximal elements that represent maximal colors for P,Q-colored graphs. For any such partially ordered set we can utilize a Fraïssé limit to construct a P,Q-colored graph for which is $\mathrm{MH}_{P,Q}$ but not $\mathrm{HH}_{P,Q}$. For this purpose, let us first define the following partially ordered set.

Definition 7. For any $n \geq 1$, F_n is the partial order consisting of an antichain of size n and a minimum element $\mathbb{0}$.

Let $\mathcal{C}_n$ the class of finite graphs with edges colored by F_n in the sense of Definition 1. This class has many interesting properties from which the most important is that it is a Fraïssé class with free amalgamation. This means that there exists a Fraïssé limit $\mathcal{R}_n$ for this class, i.e. a F_n-colored graph which is universal (contains all countable structures whose ages are contained in $\mathcal{C}_n$ as induced substructures), and ultrahomogeneous. Additionally, the following holds.

Proposition 8. *$\mathcal{R}_n$ is MH but not HH.*

This can be shown via utilization of extension property, generalizing the one defined for Rado graph [**5**], defined as follows

($\lozenge_n$) If $G_1, \ldots, G_{n+1}$ are finite disjoint subsets of $\mathcal{R}_n$, then there exists $x \in \mathcal{R}_n \setminus G_{n+1}$ such that each vertex of G_i is related to x by an edge of color c_i for $1 \leq i \leq n$, and for each vertex y of G_{n+1}, the pair xy is a nonedge.

The main role of this property is ability to ensure MH-homogeneity. While $\mathcal{R}_n$ has no vertex colors this proposition negatively answered the Problem 3.

Thanks to a claim in Lemma 6 we know that while assuming Q not being linear order under condition of Theorem 5 we know that Q is a directed set. To show that this implies existence of M which is $\mathrm{MH}_{P,Q}$ but not $\mathrm{HH}_{P,Q}$ we can use a process that is ideologically based on the following example. But first, let us provide more formal definition of a partially ordered set called a diamond.

Definition 9. Given $n \geq 2$, D_n is the partial order consisting of a finite antichain of size n, a minimum element $\mathbb{0}$, and a maximum element $\mathbb{1}$.

Having diamond formally defined, let us describe a structure M colored by D_2 via iterative process of its construction

1. Start with a countably infinite clique of color $\mathbb{1}$ partitioned into six countably infinite cliques M_x^0, M_x^1 for $x \in \{a, b, c\}$; for simplicity, use M_x to denote $M_x^0 \cup M_x^1$.
2. Add three new vertices a, b, c and connect $x \in \{a, b, c\}$ to M_x with edges of color $\mathbb{1}$.
3. Connect with color R the cliques M_a^0 to b, M_b^0 to c, M_c^0 to a, and all other edges from a clique with superindex 0 to an element of $\{a, b, c\}$ with color B. The colors are reversed for the M_x^1, that is, if M_x^0 is connected to y by color R, then M_x^1 is connected to y in color B ($x \in \{a, b, c\}, y \in \{a, b, c\} \smallsetminus \{x\}$).
4. There are no other edges in M.

Note that $\{a, b, c\}$ forms an independent set and contains all the non-edges in M. Similarly to graph $\mathcal{R}_n$ we can show that.

Proposition 10. *M is MH and not HH.*

This structure provides again an example of Q-colored graph distinguishing classes MH and HH, but more important role it plays in proof of Theorem 5.

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