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COLOURING NON-EVEN DIGRAPHS

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Abstract. A colouring of a digraph as defined by Neumann-Lara in 1982 is a vertex-colouring such that no monochromatic directed cycle exists. The minimal number of colours required for such a colouring of a digraph is defined to be its *dichromatic number*. This quantity has been widely studied in the last decades and is a natural directed analogue of the chromatic number of a graph. A digraph D is called *even* if for every 0-1-weighting of the edges it contains a directed cycle of even total weight. We show that every *non-even* digraph has dichromatic number at most 2 and an optimal colouring can be found in polynomial time. We strengthen a previously known NP-hardness result by showing that deciding whether a directed graph is 2-colourable remains NP-hard even if it contains a feedback vertex set of bounded size.

1. Introduction

The graphs we consider are simple, for digraphs however we allow antiparallel edges (*digons*). A set $X \subseteq V(D)$ is called *acyclic* if $D[X]$ is acyclic. A *feedback vertex set* F in a digraph is the complement of an acyclic vertex set, that is, F contains a vertex of each directed cycle. A *proper colouring* of a digraph D with k colours is a function $c\colon V(D) \to [k]$ such that the colour classes $c^{-1}(i)$ are acyclic for every $1 \leq i \leq k$. The *dichromatic number* $\vec{\chi}(D)$ is the smallest integer k, such that D has a proper colouring with k colours.

One of the arguably most influential problems in graph theory was the Four-Colour-Conjecture, answered positively by Appel and Haken in 1976. As a directed version of this famous theorem, the Two-Colour-Conjecture posed by Erdős and Neumann-Lara and independently by Skrekovski (see [**1**]) still stands open. A digraph D is called *oriented* if its underlying undirected graph is simple.

Conjecture 1.1 ([**12**])**.** Every oriented planar digraph D is 2-colourable.

An edge (u, v) in a digraph D is *butterfly contractible* if it is the only outgoing edge of u or the only incoming edge of v. Contracting a butterfly contractible edge and identifying parallel edges afterwards is called a *butterfly contraction* of that edge. A digraph D' is a *butterfly minor* of D if it can be obtained by butterfly contractions from a subdigraph of D. The notion of butterfly minors has been

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established in several contexts as a natural and sensible way of defining a digraph minor.

For graphs, colourings of minor-closed classes such as the planar graphs have received wide attention. One of the most intriguing problems in this area is Hadwiger's Conjecture, which has only been proved for $k \le 5$.

Conjecture 1.2 ([**6**])**.** For any $k \in \mathbb{N}$, every graph G without a K_{k+1}-minor is k-colourable.

While the above provides a simple characterisation of the largest minor-closed subclass of the k-colourable graphs, we ask the analogous question for digraphs.

Question 1.1. For $k \in \mathbb{N}$, what is the largest butterfly-minor closed class $\mathcal{D}_k$ of k-colourable digraphs?

In this paper, we initiate the study of this question by settling the first non-trivial case of $k = 2$, showing that $\mathcal{D}_2$ consists exactly of the so-called *non-even* digraphs: A digraph D is called *even* if, for every edge weighting $w\colon E(D) \to \{0,1\}$, there exists a directed cycle of even total weight in D, otherwise it is called *non-even.* For us, the following characterisation obtained by Seymour and Thomassen is crucial. An *odd bicycle* is the bidirection of an odd cycle, i.e., every (undirected) edge is replaced by a digon.

Theorem 1.1 ([**14**])**.** *A directed graph is non-even if and only if it does not contain an odd bicycle as a butterfly minor.*

Odd bicycles have dichromatic number 3 and therefore, by the above, $\mathcal{D}_2$ must be contained in the class of non-even digraphs. Our main contribution is to show the converse of this statement.

Theorem 1.2. *Let D be a non-even digraph. Then $\vec{\chi}(D) \le 2$.*

Note that this result does not directly relate to the 2-Colour Conjecture, as there are non-planar non-even digraphs and planar even digraphs. This is because the oriented planar digraphs are not closed under butterfly minors. However, the class of planar non-even digraphs is non-trivial and contains for instance the so-called *strongly planar digraphs.* For these digraphs, Theorem 1.2 yields a proof of Conjecture 1.1.

2. Matching colourings and forcing sets

There is a bijective correspondence between digraphs and bipartite graphs with a distinguished perfect matching and a bipartition: Consider an arbitrary bipartite graph $G = (A \dot\cup B, E)$ with the canonical orientation $\vec{G}(A,B)$, in which every edge starts in A and ends in B. For any perfect matching M, the digraph $\mathcal{D}(G,M)$ obtained from $\vec{G}(A,B)$ by contracting all matching edges into vertices is called the *M-direction* of G. It is not hard to reverse the described relationship to see that every digraph is an M-direction of its bipartite splitting-graph with the canonical perfect matching. In this context, it can be shown that non-even digraphs correspond exactly to the class of bipartite *Pfaffian* graphs. The latter can be characterised in terms of forbidden so-called *matching minors* of graphs, which correspond to butterfly minors of digraphs. A subgraph H of a graph G is called *conformal*, if $G - V(H)$ has a perfect matching. The *bicontraction* of a vertex of degree 2 in a graph consists of contracting both incident edges at the same time. A *matching minor* of a graph now is obtained from a conformal subgraph by repeated bicontractions of vertices of degree 2. A *bisubdivision* of a graph consists of subdividing each edge with an even number of vertices.

Theorem 2.1 ([**9**])**.** *The bipartite Pfaffian graphs are exactly those excluding $K_{3,3}$ as a matching minor. Moreover, for any bipartite graph G with a perfect matching M, G is Pfaffian if and only if $\mathcal{D}(G, M)$ is non-even.*

It is easy to see that the directed cycles in the M-direction of a bipartite graph are in bijection with the M-alternating cycles of G. The following is a natural analogue of the dichromatic number in the context of perfect matchings on graphs.

Definition 2.1. Let G be a graph with a perfect matching M. An *M-colouring* of G with $k \in \mathbb{N}$ colours is a mapping $c\colon M \to [k]$ such that for each $i \in [k]$, $c^{-1}(i) \cup (E(G) \smallsetminus M)$ does not contain an M-alternating cycle. The *M-chromatic number* $\chi(G, M)$ of a graph is the minimal number of colours required for an M-colouring.

Closely related to the above definition is the notion of a *forcing set*. Given a graph G and a perfect matching M, a partial matching $S \subseteq M$ is called *forcing* if M is the unique extension of S to a perfect matching in G. It is not hard to see that a perfect matching M of a graph is k-colourable iff there exists a partition of M into subsets $S_1, \ldots, S_k$ such that for any i, $M \smallsetminus S_i$ is forcing. Therefore, Theorem 1.2 can be rephrased as follows.

Theorem 2.2. *Every bipartite graph G with $\chi(G, M) \geq 3$ for some perfect matching M contains $K_{3,3}$ as a matching minor. Equivalently, every perfect matching of a Pfaffian bipartite graph can be partitioned into two forcing sets.*

Using the theory of decompositions of graphs into *bricks* and *braces* introduced by Lovász ([**10**]) and the theory on non-bipartite matching covered graphs developed by de Carvalho et. al. [**2, 8**], we are able to obtain similar results for classes of non-bipartite graphs.

Theorem 2.3. *If G is planar and prism-free, i.e., without a bisubdivision of $\overline{C_6}$ (the triangular prism) as a conformal subgraph, then every perfect matching can be partitioned into two forcing sets.*

The above results directly yield upper bounds of $\frac{|V(G)|}{4}$ on the minimal size of a forcing set for perfect matchings in the considered graph classes, which, to the best of our knowledge, was not known before. See [**3**] for a comprehensive survey on this topic.

3. 2-colourings of non-even digraphs

In this section we want to give a very brief sketch of the ideas involved in the proof of Theorem 1.2. The key idea is to disprove the existence of a minimal (with respect to the number of vertices) non-2-colourable non-even digraph by applying local reductions of non-even digraphs that on the one hand transport 2-colourability, while on the other hand ensure that we keep the property of being non-even. The fact that a minimal counterexample cannot be reducible is then lead to a contradiction by showing that any non-even digraph on more than two vertices can be reduced using one of the operations.

Using split operations along directed cuts and cut vertices, we start by reducing the problem to strongly 2-connected non-even digraphs. Robertson et al. [**13**] defined five different sum operations which they used to prove a generation theorem for non-even digraphs. From this we deduce the following.

Corollary 3.1. *Any strongly* 2*-connected, non-even digraph* D *on at least three vertices contains a vertex* $v \in V(D)$ *of out-degree* 2*.*

Having found such a vertex v of out-degree 2, we consider different cases concerning the local structure of the digraph around v. If v is contained in at most one digon, we apply certain deletions and a butterfly contraction to reduce the digraph, otherwise, we directly contract v and the two vertices with which v forms a digon each, into a single vertex. While this is not a standard butterfly contraction, we are able to make sense of it and show that is preserves the property of being non-even. The argument uses insights from matching theory.

The proof of Theorem 1.2 can easily be turned into a polynomial time algorithm to find a proper 2-colouring of a non-even digraph. Additionally, the work of Robertson et. al. and McCuaig [**13, 11**] imply polynomial time algorithms to recognise non-even digraphs. Hence given a digraph D we can decide whether it is non-even and then find a proper 2-colouring in polynomial time.

4. Parametrised complexity of digraph colourings

In contrast to the positive algorithmic result from the previous section, deciding whether a given digraph D has dichromatic number at most k is NP-hard for all $k \geq 2$ [**5**]. One could hope for a parametrised algorithm with respect to treewidth, as this approach works for undirected graphs by using Courcelle's Theorem [**4**]. However, the following results show that these positive results do not carry over to the world of digraphs. In fact, deciding whether a digraph is 2-colourable is NP-hard even if $\tau(D) \leq 6$, where D is the input digraph and $\tau(D)$ denotes the size of a minimum feedback vertex set in D. Consequently computing the dichromatic number is hard even if the size of a feedback vertex set or the directed treewidth is small. This strengthens the previous hardness reduction due to [**1**].

The *out-degeneracy* $d(D)$ of a digraph is defined to be the least integer x such that D and all of its subdigraphs contain a vertex of out-degree at most x. Reducing from SAT, we obtain the following.

Theorem 4.1. *For each $k \geq 2$,* Digraph k-Colouring *is* NP*-hard even if $\tau(D) \leq k+4$ and $d(D) \leq k+1$, where D is the input digraph.*

The hardness result above is relatively tight with respect to $\tau(D)$ and $d(D)$: Using a greedy strategy, it is easy to find a k-colouring if $\tau(D) \leq k-1$ or $d(D) \leq k-1$. In contrast, Theorem 4.1 excludes an $n^{f(k)}$-time algorithm under the hypothesis P$\neq$NP if we only assume $\tau(D) \leq k+4$ and $d(D) \leq k+1$ instead.

A finer analysis gives a stronger hardness result under the well-known *exponential time hypothesis* [**7**], which states that there is no subexponential-time algorithm for k-SAT

Theorem 4.2. *For each $k \geq 2$ there is some $\epsilon > 0$ such that no $2^{\epsilon n} n^{f(x,y)}$ algorithm for* Digraph k-Colouring *exists, where $x = \tau(D)$, $y = d(D)$ and f is some function, unless the ETH is false.*

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