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MIXING TIME OF THE SWAP MARKOV CHAIN AND P-STABILITY

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Abstract. The aim of this paper is to confirm that P-stability of a family of unconstrained/bipartite/directed degree sequences is sufficient for the swap Markov chain to be rapidly mixing on members of the family. This is a common generalization of every known result that shows the rapid mixing nature of the swap Markov chain on a region of degree sequences. In addition, for example, it encompasses power-law degree sequences with exponent $\gamma > 2$, and, asymptotically almost surely, the degree sequence of any Erdős-Rényi random graph $G(n,p)$ where p is bounded away from 0 and 1 by at least $\frac{5\log n}{n-1}$.

We also show that there exists a family of degree sequences which is not P-stable and its members have exponentially many realizations, yet the swap Markov chain is still rapidly mixing on them.

The first three sections of this paper are based on the preprint titled "*The mixing time of the swap (switch) Markov chains: a unified approach*", which has been uploaded to `arXiv` on March 15, 2019.

1. Introduction

An important problem in network science is to algorithmically construct typical instances of networks with predefined properties. In particular, special attention has been devoted to sampling simple graphs (i.e., loops and parallel edges are forbidden) with a given degree sequence. We study the three most common degree sequence types: the degree sequences of simple graphs which we call unconstrained degree sequences, bipartite degree sequences, and directed degree sequences.

In 1997 Kannan, Tetali, and Vempala [**12**] proposed the use of the so-called switch or *swap Markov chain* approach for uniformly sampling realizations of a degree sequence. For an unconstrained degree sequence $\mathbf{d}$ on n vertices, the swap Markov chain $(\mathbb{G}(\mathbf{d}), P(\mathbf{d}))$ is defined as follows: $\mathbb{G}(\mathbf{d})$ is the set of realizations of $\mathbf{d}$, and for two distinct realizations $G, G' \in \mathbb{G}(\mathbf{d})$, if $E(G)\triangle E(G')$ (the symmetric difference of the edge sets) is a C_4, then

$$\Pr(G \to G') = P(G, G') := \frac{1}{2\binom{n}{2}\binom{n-2}{2}}, \tag{1.1}$$

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otherwise $P(G, G') = 0$. It is well-known that this Markov chain converges to the uniform distribution on $\mathbb{G}(\mathbf{d})$. From the general theory of Markov chains it follows that the convergence is exponentially fast in terms of the number of steps taken. However, the size of $\mathbb{G}(\mathbf{d})$ can also be exponential in n, so the interesting quantity is the rate of convergence and whether it beats the growth of $|\mathbb{G}(\mathbf{d})|$ as n increases. The mixing time of a Markov chain $(\mathbb{G}, P)$ is

$$\tau_\varepsilon(\mathbb{G}, P) = \min\left\{ t : \forall X \in \mathbb{G}\ \forall t' \geq t \sum_{Y \in \mathbb{G}} \left| P^t(X, Y) - \frac{1}{|\mathbb{G}|} \right| \leq \varepsilon \right\}.$$

A Markov chain is called *rapidly mixing*, if $\tau_\varepsilon(\mathbb{G}, P) \leq \mathrm{poly}(\log|\mathbb{G}|, -\log\varepsilon)$. In his seminal work [**14**], Sinclair proved that $\tau_\varepsilon(\mathbb{G}, P) \leq \frac{1}{1-\lambda^*} \cdot (\log|\mathbb{G}| - \log\varepsilon)$, where λ^* is the second largest eigenvalue of P.

Conjecture 1.1 (the KTV conjecture [**12**])**.** The swap Markov chain is rapidly mixing for any unconstrained degree sequence.

Although Conjecture 1.1 is still open, there is a series of results that prove the rapid mixing of the swap Markov chain on various special degree sequence classes, using Sinclair's multi-commodity flow method [**14**] to bound $\frac{1}{1-\lambda^*}$. We summarize these results in a very compact way in Table 1. Most rapid mixing results on directed degree sequences can be reduced to the case of (restricted) bipartite degree sequences, as shown in [**3**].

It is not uncommon that uniformly randomly applied, small local modifications of combinatorial objects result in rapid convergence to the uniform distribution. In 1990, Jerrum and Sinclair published a very influential paper [**10**] about fast uniform generation of regular graphs and about realizations of degree sequences where no degree exceeds $\sqrt{n/2}$. To achieve this goal they applied the Markov chain they have developed in [**9**]. Informally it is known as *JS chain*, and it is sampling the perfect and near-perfect 1-factors of the Tutte gadget corresponding the degree sequence.

2. Stability and rapid mixing

The rapid mixing nature of the JS chain depends on the ratio of the number almost perfect matchings and perfect matchings [**9**], which is shown to be sufficiently low if the degree sequence $\mathbf{d}$ belongs to a *P-stable* class:

Definition 2.1. Let $\mathcal{D}$ be a set of (unconstrained, bipartite or directed) degree sequences. The class $\mathcal{D}$ is *P-stable*, if there exists a polynomial $p \in \mathbb{R}[x]$ such that for any $n \in \mathbb{N}$ and any degree sequence $\mathbf{d} \in \mathcal{D}$ on n vertices we have $\left| \bigcup_{\ell_1(\mathbf{d}', \mathbf{d}) \leq 2} \mathbb{G}(\mathbf{d}') \right| \leq p(n) \cdot |\mathbb{G}(\mathbf{d})|$.

In [**11**], the authors give a number of sufficient conditions for P-stability, and also show examples for degree sequences that are not P-stable. The "average deg. result" referred to in Table 1 refers to the following set of degree sequences:

<table>
<tr><th>unconstrained deg. seq.</th><th>bipartite deg. seq.</th><th>directed deg. seq.</th></tr>
<tr><td>regular [2]</td><td>(half-)regular [13]</td><td>regular [7]</td></tr>
<tr><td></td><td colspan="2">almost half regular [3]</td></tr>
<tr><td>$\Delta \le \frac{1}{3}\sqrt{2m}$ [8]</td><td>$\Delta \le \frac{1}{\sqrt{2}}\sqrt{m}$ [4]</td><td>$\Delta < \frac{1}{\sqrt{2}}\sqrt{m-4}$ [4]</td></tr>
<tr><td>Power-law distrib.-bound, $\gamma > 2$</td><td></td><td></td></tr>
<tr><td>$(\Delta-\delta+1)^2 \le$
$\le 4\cdot\delta(n-\Delta-1)$
[1]</td><td>$(\Delta_U-\delta_U)\cdot(\Delta_V-\delta_V) \le$
$\min(4\delta_U(|U|-\Delta_V),$
$4\delta_V(|V|-\Delta_U))$</td><td>similar to the bip. case
[4]</td></tr>
<tr><td>Erdős-Rényi $G(n,p)$
with high prob.</td><td>Bip. E.R. with edge prob.
$p, 1-p \ge 4\sqrt{\frac{2\log n}{n}}$ [4]</td><td>similar to the bip. case
[4]</td></tr>
<tr><td colspan="2">strongly stable degree sequence classes [1]</td><td></td></tr>
<tr><td colspan="3">P-stable degree sequence classes</td></tr>
</table>

Table 1. Regions for which the rapid mixing of the swap Markov chain known. Δ and δ denote the maximum and minimum degrees, respectively. Half of the sum of the degrees is m, and n is the number of vertices. The notation is similar for bipartite and directed degree sequences. Some technical conditions have been omitted. Gray text is used for previously known results.

Theorem 2.2 (Jerrum, McKay, Sinclair, [**11**, Theorem 8.3])**.** *Let $\delta = \min \mathbf{d}$, $\Delta = \max \mathbf{d}$, and $2m = \sum \mathbf{d}$. The family of unconstrained degree sequences*

$$\mathcal{D}_{JMS} := \Big\{ \mathbf{d} \in \mathbb{N}^n \ : (2m - n\delta)(n\Delta - 2m) \le (\Delta - \delta)\left((2m-n\delta)(n-\Delta-1) + (n\Delta - 2m)\delta\right) \Big\},$$

is P-stable.

Greenhill and Sfragara suggested exploring the connection between the mixing rate of the swap Markov chain and stable degree sequences [**8**, Subsection 1.1]. The first such result is due to Amanatidis and Kleer:

Theorem 2.3 ([**1**])**.** *The swap Markov chain is rapidly mixing on strongly stable unconstrained and bipartite degree sequence classes.*

A set of degree sequences $\mathcal{D}$ is strongly stable, if there exists a constant C, such that for any degree sequence $\mathbf{d}'$ for which $\ell_1(\mathbf{d}', \mathbf{d}) \le 2$ for some $\mathbf{d} \in \mathcal{D}$, for any $G' \in \mathbb{G}(\mathbf{d}')$ there exists $G \in \mathbb{G}(\mathbf{d})$ such that $|E(G') \triangle E(G)| \le C$. Strong stability directly implies P-stability; in fact, the proof of Theorem 2.2 in [**11**] implicitly shows that $\mathcal{D}_{\mathrm{JMS}}$ is a strongly stable class. As far as we known, Theorem 2.3

implies all of the previously known rapid mixing results on unconstrained and bipartite degree sequences (see Table 1).

Our main result is extending the techniques of [**13, 3**] to unconstrained, bipartite, and directed degree sequences with a unified machinery:

Theorem 2.4. *The swap Markov chain is rapidly mixing on P-stable unconstrained, bipartite, and directed degree sequence classes.*

Remark 2.5. On realizations of directed degree sequences, switching the orientation of edges along directed circuit of length 6 is allowed, too. If this operation is not permitted, the swap Markov chain is not even irreducible (connected) for directed degree sequences.

The unified framework in which we prove the theorem requires only minimal branching between the three different types of degree sequences. The theoretical novelty of this result is twofold. We scaled up previous results both

- vertically (power of machinery) to P-stable degree sequence classes, and
- horizontally (applicability of machinery) to directed degree sequences.

To our knowledge, Theorem 2.4 applies to every region of degree sequences where the rapid mixing of the swap Markov chain has been known (Table 1).

3. Applications of Theorem 2.4

There are two interesting consequences of Theorem 2.4 concerning popular random graph models. The next theorem follows via Hoeffding's inequality and Theorem 2.2.

Theorem 3.1. *The degree sequence of an unconstrained Erds-Rnyi random graph with edge probability p bounded away from 0 and 1 by at least* $\frac{5\log n}{n-1}$ *is asymptotically almost surely P-stable (as* $n \to \infty$*).*

Informally, the theorem is a consequence of the fact that the degrees in an Erdős-Rényi random graph are tightly concentrated around their expected value.

Scale-free networks whose (unconstrained) degree sequences follow a power-law are immensely popular in the network science community. A set of degree sequences $\mathcal{D}$ is *power-law distribution-bounded* with parameter γ, if there exists a constant $C > 0$ such that for all n and $\mathbf{d} \in \mathcal{D}$ on n vertices, we have $|\{k : \mathbf{d}(k) \geq i\}| \leq Cn\sum_{j\geq i} j^{-\gamma}$ for any $i \geq 1$. The definition of *power-law density-bound* is similar, but stricter, because it requires that $|\{k : \mathbf{d}(k) = i\}| \leq Cni^{-\gamma}$ for any $i \geq 1$. In degree distributions of empirical networks following a power-law, the parameter γ is usually between 2 and 3 (see [**5**]).

As stated in Table 1, Greenhill and Sfragara [**8**] showed that the swap Markov chain is rapidly mixing on power-law density-bounded degree sequences with $\gamma > 2.5$. This is based on Gao and Wormald's enumeration of several heavy-tailed degree sequences [**5**]. In particular, they estimate the number of realizations of degree sequences that are

(1) power-law *density-bounded* with $\gamma > 2.5$,
(2) power-law *distribution-bounded* with $\gamma > 1 + \sqrt{3} \approx 2.732$.

The formula in [**5**] enumerating the realizations of degree sequences obeying either (1) or (2) directly implies P-stability, thus our Theorem 2.4 applies. Gao and Wormald [**6**] also claim that degree sequences obeying a power-law distribution-bound with $\gamma > 2$ are P-stable, therefore:

Theorem 3.2 (Follows from Theorem 2.4 and [**6**])**.** *The swap Markov chain is rapidly mixing on degree sequences satisfying a power-law distribution-bound for any $\gamma > 2$.*

It is not known whether this set of degree sequences is strongly stable [**1**].

4. Beyond P-stability

The notion of P-stability is a natural obstacle on the rapid mixing of the swap Markov chain [**10, 11**]. Let us define the following bipartite degree sequences:

$$\mathbf{h}(n) := \begin{pmatrix} n & n-1 & n-2 & n-3 & \cdots & 3 & 2 & 1 \\ n & n-1 & n-2 & n-3 & \cdots & 3 & 2 & 1 \end{pmatrix}$$

$$\mathbf{g}(n) := \begin{pmatrix} n-1 & n-1 & n-2 & n-3 & \cdots & 3 & 2 & 1 \\ n-1 & n-1 & n-2 & n-3 & \cdots & 3 & 2 & 1 \end{pmatrix}$$

Trivially, $|\mathbb{G}(\mathbf{h})| = 1$. The unique realization of $\mathbf{h}(n)$ is called the *half-graph* $H(n)$. It is easy to see that the number of realizations of $\mathbf{g}(n)$ satisfies

$$|\mathbb{G}(\mathbf{g}(n))| = 2|\mathbb{G}(\mathbf{g}(n-1))| + \sum_{i=1}^{n-2} |\mathbb{G}(\mathbf{g}(i))|.$$

Solving the linear recursion gives

Lemma 4.1. $|\mathbb{G}(\mathbf{g}(n))| = \left(\frac{3+\sqrt{5}}{2}\right)^{n-o(n)}$.

Thus the degree sequences of half-graphs are not P-stable, even though it is trivial to sample them. A slightly more complicated counting argument shows that

Lemma 4.2. *The set $\{\mathbf{g}(n) : n \geq 3\}$ is not P-stable.*

However, there is a lot of structure in the realizations of $\mathbf{g}(n)$. First, observe that $H(n)$ has a very natural planar drawing, as demonstrated on Figure 1. Suppose $G \in \mathbb{G}(\mathbf{g}(n))$, and take $\nabla = E(G) \triangle E(H(n))$.

Observe, that ∇ is composed of two parts: $E(G) \setminus E(H(n))$ (red edges) and $E(H(n)) \setminus E(G)$ (blue edges). The red and blue degrees of vertices in ∇ are all equal, except at x_n and y_n, where the blue degree is one larger than the red degree. This means that ∇ can be decomposed into a red-blue alternating path and some red-blue alternating circuits. However, a red-blue alternating circuit is an alternating circuit in $H(n)$, so we could exchange the edges and non-edges in it while preserving the degree sequence. But this is clearly impossible, as $\mathbf{h}(n)$ has exactly one realization.

Thus ∇ is an alternating path in $H(n)$ connecting its degree n vertices. In fact, we have shown, that

Lemma 4.3. *The realizations of* $\mathbf{g}(n)$ *are in a* 1*-to-*1 *correspondence with the left-to-right paths between the degree n vertices of* $H(n)$ *in* Figure 1.

This bijection allows us to design an efficient multi-commodity flow on the Markov graph on $\mathbb{G}(\mathbf{g}(n))$ (two realizations are joined by an edge iff they can be transformed into each other via one swap). Thus we obtain (possibly the first) non-trivial rapid mixing result of the swap Markov chain on non-stable degree sequences using Sinclair's method [**14**].

Theorem 4.4. *The swap Markov chain is rapidly mixing on degree sequences in* $\{\mathbf{g}(n) : n \geq 3\}$.

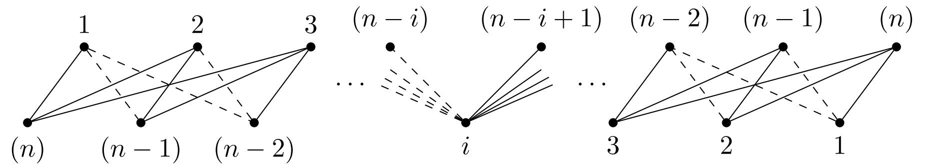

Figure 1. The half-graph $H(n)$. The vertices are drawn in descending order and spaced regularly. In the upper color class the vertices are drawn in the same manner except their order is ascending and they are shifted to the right by 1.5 distance between two consecutive vertices. This way, for every vertex in the lower color class, the edges are going right and non-edges (dashed lines) are going left, so an alternating path goes left to right.

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