Acta Math. Univ. Comenianae
Vol. LXXXVIII, 3 (2019), pp. 891–895

CHARACTERIZATION OF GENERALISED PETERSEN GRAPHS THAT ARE KRONECKER COVERS

M. KRNC AND T. PISANSKI

ABSTRACT. The family of generalised Petersen graphs $G(n,k)$, introduced by Coxeter (1950) and named by Watkins (1969) is a family of cubic graphs formed by connecting the vertices of a regular polygon to the corresponding vertices of a star polygon. The Kronecker cover $\mathrm{KC}(G)$ of a simple undirected graph G is a special type of bipartite covering graph of G, isomorphic to the direct (tensor) product of G and K_2. We characterize all generalised Petersen graphs that are Kronecker covers, and describe the structure of their respective quotients. We observe that some of such quotients are again generalised Petersen graphs, and describe all such pairs.

1. INTRODUCTION

The *generalised Petersen graphs*, introduced by [**4**] and named by [**15**], form a very interesting family of trivalent graphs that can be described by only two integer parameters. They include Hamiltonian and non-Hamiltonian graphs, bipartite and non-bipartite graphs, vertex-transitive and non-vertex-transitive graphs, Cayley and non-Cayley graphs, arc-transitive graphs and non-arc-transitive graphs, graphs of girth $3, 4, 5, 6, 7$ or 8. Their generalization to I-graphs does not introduce any new vertex-transitive graphs but it contains also non-connected graphs and has in special cases unexpected symmetries [**2**]. For further properties of I-graphs also see [**8, 12**].

Following the notation of [**15**], for given integers n and $k < \frac{n}{2}$, we can define a generalised Petersen graph $G(n,k)$ as a graph on vertex-set $\{u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}\}$. The edge-set may be naturally partitioned into three equal parts (note that all subscripts are assumed modulo n): the edges $E_O(n,k) = \{u_i u_{i+1}\}_{i=0}^{n-1}$ form the *outer rim*, inducing a cycle of length n; the edges $E_I(n,k) = \{v_i v_{i+k}\}_{i=0}^{n-1}$ form the *inner rims*, inducing $\gcd(n,k)$ cycles of length $\frac{n}{\gcd(n,k)}$; and the edges $E_S(n,k) = \{u_i v_i\}_{i=0}^{n-1}$, also called *spokes*, which induce a perfect matching in

Received May 31, 2019.
2010 *Mathematics Subject Classification.* Primary 57M10, 05C10, 05C25.
The authors gratefully acknowledge the European Commission for funding the InnoRenew CoE project (Grant Agreement #739574) under the Horizon2020 Widespread-Teaming program and the Republic of Slovenia (Investment funding of the Republic of Slovenia and the European Union of the European regional Development Fund).

$G(n,k)$. Hence the edge-set may be defined as $E(G(n,k)) = E_O(n,k) \cup E_I(n,k) \cup E_S(n,k)$.

Various structural aspects of the mentioned family have been pointed out. Examples include identifying generalised Petersen graphs that are Hamiltonian [**1**] or Cayley [**11, 13**], or isomorphic [**8, 12, 14**], or finding their automorphism group [**5**]. Also, a related generalization to I-graphs has been introduced in the Foster census [**3**], and further studied by [**2**].

The theory of covering graphs became one of the most important and successful tools of algebraic graph theory. It is a discrete analog of the well known theory of covering spaces in algebraic topology. In general, covers depend on the values called voltages assigned to the edges of the graphs. Only in some cases the covering is determined by the graph itself. One of such cases is the recently studied *clone cover* [**10**]. The other, more widely known case is the Kronecker cover.

The *Kronecker cover* $\mathrm{KC}(G)$ (also called bipartite or canonical double cover) of a simple undirected graph G is a bipartite covering graph with twice as many vertices as G. Formally, $\mathrm{KC}(G)$ is defined as a tensor product $G \times K_2$, i.e. a graph on a vertex-set $V(\mathrm{KC}(G)) = \{v', v''\}_{v \in V(G)}$, and the edge-set $E(\mathrm{KC}(G)) = \{u'v'', u''v'\}_{uv \in E(G)}$. For $H = \mathrm{KC}(G)$, we also say that G is *a quotient* of H. Some recent work on Kronecker covers includes [**6**] and [**9**].

In this paper, we study the family of generalised Petersen graphs in conjunction with the Kronecker cover operation. Namely, in the next section we state our main theorem characterizing all generalised Petersen graphs that are Kronecker covers, and describing the structure of their corresponding quotient graphs. We conclude the paper with some remarks and directions for possible future research.

2. Main result

In order to state the main result we need to introduce the graph $\mathcal{H}$ and two 2-parametric families of cubic, connected graphs.

Let $\mathcal{H}$ be the graph defined by the following procedure: Take the Cartesian product $K_3 \square P_3$, remove the edges of the middle triangle, add a new vertex and connect it to all three 2-vertices. Note that the graph $\mathcal{H}$ is mentioned in [**9**] and is depicted in Figure 1.

As shown in [**9**], the Desargues graph $G(10,3)$ is the Kronecker cover of both $G(5,2)$ and $\mathcal{H}$. Note that in Figure 1 the edge-colored subgraphs of $G(5,2)$ and $\mathcal{H}$ lift to the edge-colored subgraphs of $G(10,3)$, respectively.

In a Hamiltonian cubic graph, the vertices can be arranged in a cycle, which accounts for two edges per vertex. The third edge from each vertex can then be described by how many positions clockwise (positive) or counter-clockwise (negative) it leads. This approach of describing Hamiltonian cubic graph, named by developers Lederberg, Coxeter and Frucht, is called the *LCF notation*, and is defined by the sequence $[a_0, a_1, \ldots, a_{n-1}]$ of numbers of positions, starting from an arbitrarily chosen vertex. To state our results, we only use a special type of such LCF-representable graphs, namely $C^+(n,k)$ and $C^-(n,k)$, which we define below.

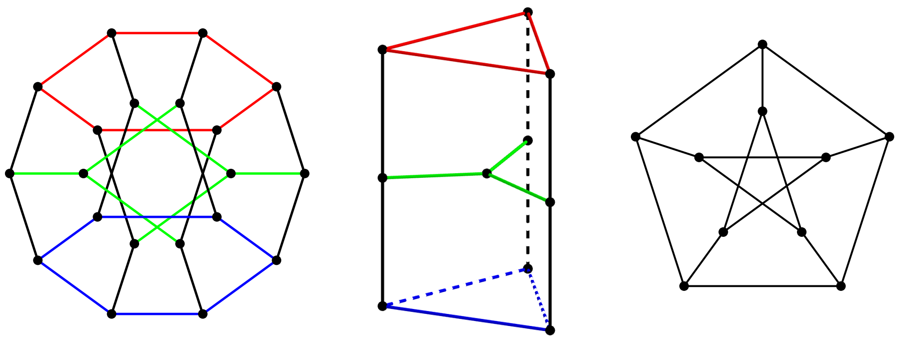

Figure 1. The Desargues graph and both its quotients; $\mathcal{H}$ and the Petersen graph.

Definition 1. Assuming all numbers are modulo n, define graphs

$$C^+(n,k) = \left[\frac{n}{2}, \frac{n}{2} + (k-1), \frac{n}{2} + 2(k-1), \dots, \frac{n}{2} + (n-1)(k-1)\right],$$

and similarly

$$C^-(n,k) = \left[\frac{n}{2}, \frac{n}{2} - (k+1), \frac{n}{2} - 2(k+1), \dots, \frac{n}{2} - (n-1)(k+1)\right].$$

Note that throughout the paper, any instance of the definition above is used with such values (n,k), so that the corresponding graph is well defined. In [**9**] it was proven that $G\,(10,3)$ is the Kronecker cover of two non-isomorphic graphs. Here we prove among other things that this is the only generalised Petersen graph that is a multiple Kronecker cover. Every other generalised Petersen graph is either a Kronecker cover of a single graph or it is not a Kronecker cover at all. More precisely;

Theorem 1. *Among the members of the family of generalised Petersen graphs, $G\,(10,3)$ is the only graph that is the Kronecker cover of two non-isomorphic graphs, the Petersen graph and the graph $\mathcal{H}$. For any other $G \simeq G\,(n,k)$, the following holds:*

a) If $n \equiv 2 \pmod 4$ and k is odd, G is a Kronecker cover. In particular
- *a_1) if $4k < n$, the corresponding quotient graph is $G\left(\frac{n}{2}, k\right)$, and*
- *a_2) if $n < 4k < 2n$ the quotient graph is $G\left(\frac{n}{2}, \frac{n}{2} - k\right)$.*

b) If $n \equiv 0 \pmod 4$ and k is odd, G is a Kronecker cover if and only if $n \mid \frac{k^2-1}{2}$ and $k < \frac{n}{2}$. Moreover,
- *b_1) if $k = 4t+1$ the corresponding quotient is $C^+(n,k)$ while*
- *b_2) if $k = 4t+3$ the quotient is $C^-(n,k)$.*

c) Any other generalised Petersen graph is not a Kronecker cover.

For $k = 1$ and even n each $G(n,1)$ is a Kronecker cover. If $n = 4t$ case b_1) applies and the quotient graph is the Möbius ladder M_n (see [**7**]). For $G(4,1)$ the quotient is $K_4 = M_4$. Similarly, the 8-sided prism $G(8,1)$ is the Kronecker cover of M_8. In case $n = 4t + 2$ the case a_1) applies and the quotient is $G(n/2, 1)$. For instance, the 6-sided prism is the Kronecker cover of the 3-sided prism.

3. Concluding remarks and future work

In this paper, we classified parameters (n, k) such that $G\,(n, k)$ is a Kronecker cover of some graph, and described the corresponding quotients. From our main result it easily follows:

Corollary 2. $\mathrm{KC}\,(G\,(n, k))$ *is itself a generalised Petersen graph if and only if* n *is odd.*

Graphs $\mathrm{KC}\,(G\,(n, k))$ that are not generalised Petersen graphs, in other words if n is odd, fall into two known classes, depending on the parity of k. If k is odd, we have $\mathrm{KC}\,(G\,(n, k)) = 2G\,(n, k)$. It would be interesting to investigate the family of graphs $\mathrm{KC}\,(G\,(n, k))$ with both n and k even. The smallest case is depicted in Figure 2. This is the Kronecker cover of the Dürer graph $G\,(6, 2)$.

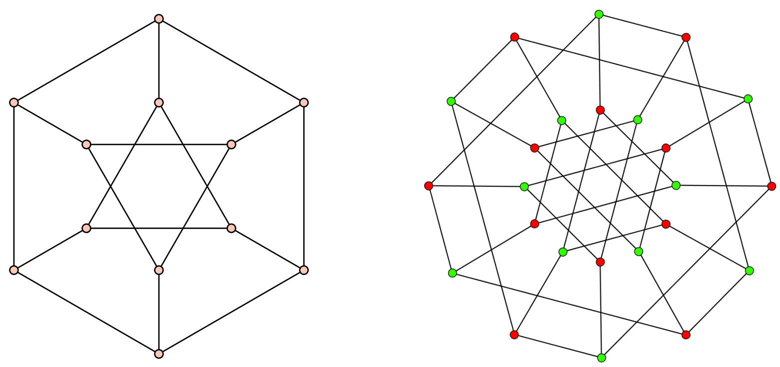

Figure 2. The Dürer graph $G\,(6, 2)$ and its Kronecker cover $KC(G\,(6, 2))$ with proper vertex two-coloring.

Acknowledgment. We would like to thank A. Hujdurović and A. Žitnik for helpful comments and discussions and Gábor Gévay for the drawing of the Kronecker cover of the Dürer graph. The authors gratefully acknowledge the European Commission for funding the InnoRenew CoE project (Grant Agreement #739574) under the Horizon2020 Widespread-Teaming program and the Republic of Slovenia (Investment funding of the Republic of Slovenia and the European Union of the European regional Development Fund). The second author gratefully acknowledges financial support of the Slovenian Research Agency, ARRS, research program no. P1-0294. The extended version of this paper was recently submitted for publication to another journal, and is currently pending a review.

References

1. Alspach B., *The classification of hamiltonian generalized Petersen graphs*, J. Comb. Theory Ser. B **34** (1983), 293–312.
2. Boben M., Pisanski T. and Žitnik A., *I-graphs and the corresponding configurations*, J. Combin. Des. **13** (2005), 406–424.
3. Bouwer I., Chernoff W., Monson B. and Star Z., *The Foster Census*, Charles Babbage Research Centre, Winnipeg.
4. Coxeter, H. et al., *Self-dual configurations and regular graphs*, Bull. Amer. Math. Soc. **56** (1950), 413–455.
5. Frucht R., Graver J. E. and Watkins M. E., *The groups of the generalized Petersen graphs*, in: Math. Proc. Camb. Philos. Soc., vol. 70, Cambridge Univ. Press, 211–218.
6. Gévay G. and Pisanski T., *Kronecker covers, V-construction, unit-distance graphs and isometric point-circle configurations*, Ars Math. Contemp. **7** (2014), 317–336.
7. Guy R. K. and Harary F., *On the möbius ladders*, Can. Math. Bull. **10** (1967), 493–496.
8. Horvat B., Pisanski T. and Žitnik A., *Isomorphism checking of I-graphs*, Graphs Combin. **28** (2012), 823–830.
9. Imrich W. and Pisanski T., *Multiple Kronecker covering graphs*, European J. Combin. **29** (2008), 1116–1122.
10. Malnič A., Pisanski T. and Žitnik A., *The clone cover*, Ars Math. Contemp. **8** (2014), 95–113.
11. Nedela R. and Škoviera M., *Which generalized Petersen graphs are Cayley graphs?*, J. Graph Theory **19** (1995), 1–11.
12. Petkovšek M. and Zakrajšek H., *Enumeration of I-graphs: Burnside does it again*, Ars Math. Contemp. **2** (2009), 241–262.
13. Saražin M. L., *A note on the generalized Petersen graphs that are also Cayley graphs*, J. Comb. Theory Ser. B **69** (1997), 226–229.
14. Steimle A. and Staton W., *The isomorphism classes of the generalized Petersen graphs*, Discr. Math. **309** (2009), 231–237.
15. Watkins M. E., *A theorem on Tait colorings with an application to the generalized Petersen graphs*, J. Comb. Theory **6** (1969), 152–164.

M. Krnc, Faculty of Mathematics, Natural Sciences and Information Technologies University of Primorska, Koper, Slovenia,
e-mail: `matjaz.krnc@upr.si`

T. Pisanski, Faculty of Mathematics, Natural Sciences and Information Technologies University of Primorska, Koper, Slovenia,
e-mail: `pisanski@upr.si`