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RECONFIGURATION GRAPH FOR VERTEX COLOURINGS OF WEAKLY CHORDAL GRAPHS

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ABSTRACT. The reconfiguration graph $R_k(G)$ of the k-colourings of a graph G contains as its vertex set the k-colourings of G and two colourings are joined by an edge if they differ in colour on just one vertex of G.

We show that for each $k \geq 3$ there is a k-colourable weakly chordal graph G such that $R_{k+1}(G)$ is disconnected. We also introduce a subclass of k-colourable weakly chordal graphs which we call k-colourable compact graphs and show that for each k-colourable compact graph G on n vertices, $R_{k+1}(G)$ has diameter $O(n^2)$. We show that this class contains all k-colourable co-chordal graphs and when $k = 3$ all 3-colourable $(P_5, \overline{P_5}, C_5)$-free graphs. We also mention some open problems.

1. Introduction

Let G be a graph, and let k be a non-negative integer. A k-colouring of G is a function $f : V(G) \to \{1, \dots, k\}$ such that $f(u) \neq f(v)$ whenever $(u, v) \in E(G)$. The reconfiguration graph $R_k(G)$ of the k-colourings of G has as vertex set the set of all k-colourings of G and two vertices of $R_k(G)$ are adjacent if they differ on the colour of exactly one vertex (the change of the colour is the so called *colour switch*). For a positive integer ℓ, the *ℓ-colour diameter* of a graph G is the diameter of $R_\ell(G)$.

In the area of reconfigurations for colourings of graphs, one focus is to determine the complexity of deciding whether two given colourings of a graph can be transformed into one another by a sequence of recolourings (that is, to decide whether there is a path between these two colourings in the reconfiguration graph); see, for example, [**8, 7, 5, 3**]. Another focus is to determine the diameter of the reconfiguration graph in case it is connected or the diameter of its components if it is disconnected [**2, 6, 1, 4, 10**]. We refer the reader to [**13, 12**] for excellent surveys on reconfiguration problems.

In this note, we continue the latter line of study of reconfiguration problems. In Section 3, we shall show that the $(k+1)$-colour diameter of k-colourable weakly chordal graphs can be infinite. On the positive side, in Section 4, we shall consider

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two specific subclasses of k-colourable perfect graphs and show that their $(k+1)$-colour diameter is quadratic in the order of the graph.

We omit all proofs due to space restrictions.

2. Preliminaries

For a graph $G = (V, E)$ and a vertex $u \in V$, let $N_G(u) = \{v : uv \in E\}$. A *separator* of a graph $G = (V, E)$ is a set $S \subset V$ such that $G - S$ has more connected components than G. If two vertices u and v that belong to the same connected component in G are in two different connected components of $G - S$, then we say that S *separates* u and v. A *chordless path* P_n of length $n-1$ is the graph with vertices $v_1, \ldots, v_n$ and edges v_iv_{i+1} for $i = 1, \ldots, n-1$. It is a *cycle* C_n of length n if the edge v_1v_n is also present.

The *complement* of G is denoted $\overline{G} = (V, \overline{E})$. It is the graph on the same vertex set as G and there is an edge in G between two vertices u and v if and only if there is no edge between u and v in $\overline{G}$. A set of vertices in a graph is *anticonnected* if it induces a graph whose complement is connected. A *clique* or a *complete graph* is a graph where every pair of vertices is joined by an edge. The size of a largest clique in a graph G is denoted $\omega(G)$. The *chromatic number* $\chi(G)$ of a graph G is the least integer k such that G is k-colourable.

A graph G is *perfect* if $\omega(G') = \chi(G')$ for every (not necessarily proper) subgraph G' of G. A *hole* in a graph is a cycle of length at least 5 and an *antihole* is the complement of a hole. A graph is perfect if it is (odd hole, odd antihole)-free [**9**]. A graph is *weakly chordal* if it is (hole, antihole)-free. A graph is *co-chordal* if it is ($\overline{C_4}$, anti-hole)-free. Every weakly chordal graph is perfect. Every co-chordal graph and every $(P_5, \overline{P_5}, C_5)$-free graph is weakly chordal.

A *2-pair* of a graph G is a pair $\{x, y\}$ of nonadjacent distinct vertices of G such that every chordless path from x to y has length 2. We often use the following well-known lemma:

Lemma 2.1 (Hayward et al. [**11**]). *A graph G is weakly chordal graph if and only if every subgraph of G is either a complete graph or it contains a 2-pair.*

3. Weakly chordal graphs

In this section, we establish the following result.

Theorem 3.1. *For each $k \geq 3$ there exists a k-colourable weakly chordal graph G_k such that $R_{k+1}(G_k)$ is disconnected.*

The graph G_k is depicted in Figure 1.

In other words, Theorem 3.1 states that for each $k \geq 3$ the $(k+1)$-colour diameter of k-colourable weakly chordal graphs can be infinite. It is worth mentioning that the case $k = 2$ is already known [**2**] as the class of 2-colourable weakly chordal graphs is precisely the class of chordal bipartite graphs. It is also worth mentioning that Bonamy, Johnson, Lignos, Patel and Paulusma [**2**] asked whether the $(k+1)$-colour diameter of k-colourable perfect graphs is connected. This was

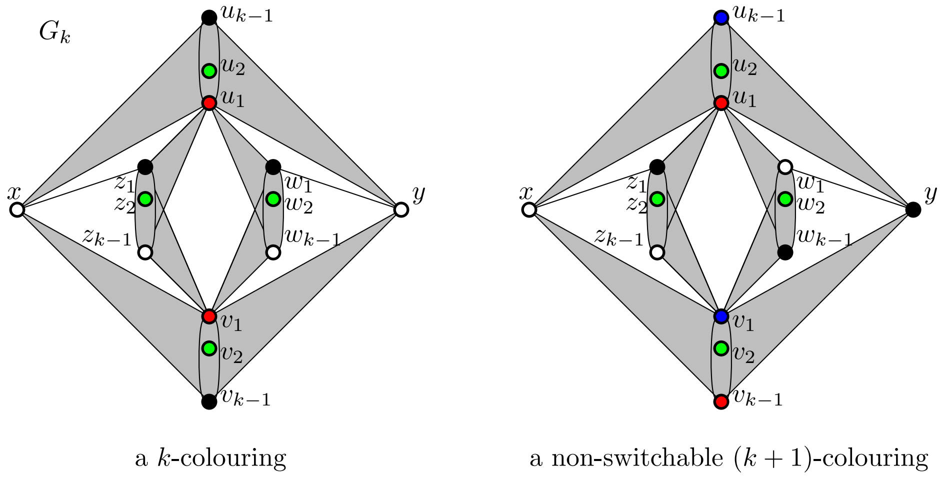

Figure 1. The graph G_k. Each gray area corresponds to a clique.

answered negatively in [**1**] – the counterexample consists of a complete bipartite graph minus a matching. Our Theorem 3.1 thus strengthens this counterexample.

4. Quadratic diameter

In this section, we introduce a subclass of k-colourable weakly chordal graphs that we call k-colourable compact graphs. We show in Theorem 4.1 that for each k-colourable compact graph G on n vertices the diameter of $R_{k+1}(G)$ is $O(n^2)$. We then show in Lemma 4.1 that k-colourable co-chordal graphs are k-colourable compact and in Lemma 4.2 that 3-colourable $(P_5, \overline{P_5}, C_5)$-free graphs are 3-colourable compact.

For a 2-pair $\{u, v\}$ of a weakly chordal graph G, let $S(u, v) = N_G(u) \cap N_G(v)$. Note that, by the definition of a 2-pair, $S(u, v)$ is a separator of G that separates u and v. Let C_v denote the component of $G \setminus S(u, v)$ that contains the vertex v.

Definition 4.1. A weakly chordal graph G is said to be *compact* if every subgraph H of G either

(i) is a complete graph, or
(ii) contains a 2-pair $\{x, y\}$ such that $N_H(x) \subseteq N_H(y)$, or
(iii) contains a 2-pair $\{x, y\}$ such that $C_x \cup S(x, y)$ induces a clique on at most three vertices.

Theorem 4.1. *Let k be a positive integer, and let G be a k-colourable compact graph on n vertices. Then $R_{k+1}(G)$ has diameter $O(n^2)$.*

Lemma 4.1. *Every k-colourable co-chordal graph is compact.*

Lemma 4.2. *Every 3-colourable $(P_5, \overline{P_5}, C_5)$-free graph is compact.*

We are aware the concept of compact graphs does not fit tight with the class of $(P_5, \overline{P_5}, C_5)$-free graphs, as some of these graphs need not to be k-colourable compact graphs for $k \geq 4$. An example of such graph H for $k = 4$ is depicted in Figure 2.

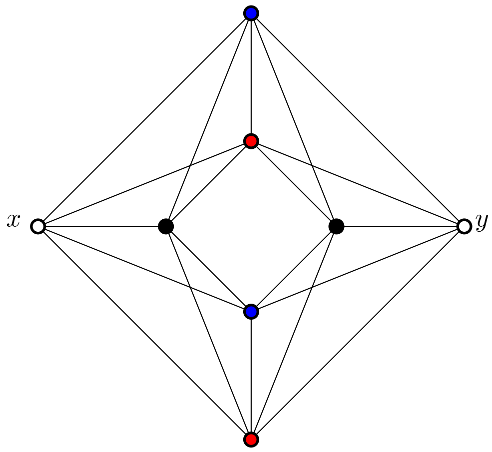

Figure 2. A $(P_5, \overline{P_5}, C_5)$-free 4-colourable graph H that is not compact.

Due to symmetries of the graph H it suffices without loss of generality to consider only the 2-pair $\{x, y\}$ as other 2-pairs could be mapped onto $\{x, y\}$ by an automorphism of H. Observe that this 2-pair violates the conditions of the Definition 4.1 for H to be 4-colourable compact.

Any choice of five vertices from H would contain two vertices joined by a horizontal or a vertical edge, and such edge cannot be extended to an induced P_3, hence H is also P_5-free. Also, such choice of five vertices would contain two opposite vertices either of the inner C_4 or from the outer one, like the vertices x and y. As such two vertices form an 2-pair, H contains no C_5. Finally, H has only two induced C_4 and neither could be completed by any fifth vertex to a $\overline{P_5}$.

5. Concluding remarks

We end this note with two open problems.

Problem 1. For which integer $\ell > k+1$ is the ℓ-colour diameter of k-colourable weakly chordal graphs connected?

Problem 2. Is the $(k+1)$-colour diameter of k-colourable $(P_5, \overline{P_5}, C_5)$-free graphs quadratic for each $k \geq 4$?

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