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ON THE MAXIMUM NUMBER OF ODD CYCLES IN GRAPHS WITHOUT SMALLER ODD CYCLES

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ABSTRACT. We prove that for each odd integer $k \geq 7$, every graph on n vertices without odd cycles of length less than k contains at most $(n/k)^k$ cycles of length k. This generalizes the previous results on the maximum number of pentagons in triangle-free graphs, conjectured by Erdős in 1984, and asymptotically determines the generalized Turán number $\mathrm{ex}(n, C_k, C_{k-2})$ for odd k. In contrast to the previous results on the pentagon case, our proof is not computer-assisted.

1. INTRODUCTION

The problem of determining Turán numbers $\mathrm{ex}(n, H)$, defined as the maximum number of edges in an H-free graph on n vertices, is one of the most classical problems in graph theory. The original Turán Theorem [**16**] solves it for cliques and the Erdős-Stone-Simonovits Theorem [**4**] determines the asymptotic behavior of $\mathrm{ex}(n, H)$ for any other non-bipartite graph H. The remaining bipartite case contains many interesting and longstanding open problems, as well as important results, see for example surveys by Füredi and Simonovits [**6**], Sidorenko [**15**], or, in the case of cycles, the survey by Verstraëte [**17**].

Recently a lot of attention is attracting a generalized problem of determining the value of $\mathrm{ex}(n, T, H)$, the maximum possible number of copies of a graph T in any H-free graph on n vertices. Systematic studies of this problem were initiated by Alon and Shikhelman [**1**], but some specific cases, in particular for T and H being cycles, were considered earlier. Bollobás and Győri [**2**] proved that $\mathrm{ex}(n, C_3, C_5) = \Theta(n^{3/2})$, Győri and Li [**10**] extended this result to obtain bounds for $\mathrm{ex}(n, C_3, C_{2k+1})$, which were later improved by Alon and Shikhelman [**1**] and by Füredi and Özkahya [**5**]. Recently, Gishboliner and Shapira [**8**] proved the correct order of magnitude of $\mathrm{ex}(n, C_k, C_\ell)$ for each k and ℓ and independently

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Gerbner, Győri, Methuku and Vizer [**7**] for all even cycles, together with the tight asymptotic value of $\mathrm{ex}(n, C_4, C_{2k})$.

Much earlier, in 1984, Erdős conjectured [**3**] that $\mathrm{ex}(n, C_5, C_3) \le (n/5)^5$ and the equality holds for n divisible by 5 at the balanced blow-up of a 5-cycle. It was proved using the method of flag algebras independently by Grzesik [**9**] and Hatami, Hladký, Kráľ, Norine, and Razborov [**11**]. Later, also using the flag algebras method, Lidický and Pfender [**13**] showed that the balanced blow-ups of C_5 are the only graphs that maximize the number of copies of C_5 for each n with the exception of n equal to 8, for which there are more extremal graphs, as was first pointed out by Michael [**14**].

2. Main result

Here, we prove a generalization of the above result by showing the following theorem.

Theorem 1. *For each odd integer $k \ge 7$, any graph on n vertices without odd cycles of length less than k contains at most $(n/k)^k$ cycles of length k. Moreover, the balanced blow-up of a k-cycle is the only graph attaining this maximum.*

It is worth mentioning that, in contrast to the previous results on the pentagon case, our proof does not use flag algebras and is not computer-assisted. However, it does not extend to $k = 5$, since it requires that the graph does not contain any 5-cycles.

By the standard application of the graph removal lemma, our result (Theorem 1) determines the tight asymptotic value of $\mathrm{ex}(n, C_k, C_{k-2})$ for all odd k unknown before.

Corollary 2. *For any odd integer $k \ge 7$, $\mathrm{ex}(n, C_k, C_{k-2}) = (n/k)^k + o(n^k)$.*

The proof of the main result is based on the method developed by Kráľ, Norin, and Volec [**12**]. We finish with an overview of the proof.

Fix an odd integer $k \ge 7$ and let G be any graph without C_ℓ for all odd ℓ between 3 and $k-2$. Since there are no smaller odd cycles than k, each k-cycle in G is induced.

The main idea is to define weights on sequences of vertices of length k. For each prefix subsequence we define a set of vertices extending this subsequence in a "good way" (what this means it depends on the length of the subsequence; for instance, it may mean that these vertices in some special order form an induced path). We define then the weight of some sequence on k vertices by taking the inverse of the product of sizes of these sets for all prefix subsequences — assuming that this product is non-zero. In particular, if the weight of some sequence is non-zero, then the vertices of this sequence induce some cycle of length k and each such cycle is induced by exactly $2k$ different sequences (obtained by cyclic shifts and reversing the order) with non-zero weight.

It is straightforward that the sum of weights of all sequences is at most 1. Hence in order to prove the theorem it would be enough to show a proper lower bound for these weights. However, for any such sequence in the balanced blow-up of a k-cycle about $1/k$ fraction of the vertices do not contribute at all to its weight, and in arbitrary graph the minimum possible weight may be much smaller than the average weight. To handle this problem, we consider instead the sum of weights over all cyclic shifts of some fixed sequence — in this way we average the contribution of individual vertices and are able to get the right bound.

Uniqueness of the extremal construction follows from the fact that the only graph that can satisfy the considered inequalities as equalities is the balanced blow-up of a k-cycle.

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