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A BROOKS-LIKE RESULT FOR GRAPH POWERS

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Abstract. Coloring a graph G consists in finding an assignment of colors $c\colon V(G) \to \{1, \ldots, p\}$ such that any pair of adjacent vertices receive different colors. The minimum integer p such that a coloring exists is called the chromatic number of G, denoted by $\chi(G)$. We investigate the chromatic number of powers of graphs, i.e. the graphs obtained from a graph G by adding an edge between every pair of vertices at distance at most k. For $k = 1$, Brooks' theorem states that every graph of maximum degree $\Delta \geq 3$ excepted the clique on $\Delta + 1$ vertices can be colored using Δ colors (i.e., one color less than the naive upper bound). For $k \geq 2$, a similar result holds: excepted for Moore graphs, the naive upper bound can be lowered by 2. We prove that for $k \geq 3$ and for every Δ, we can actually spare $k - 2$ colors, excepted for a finite number of graphs.

1. Introduction

Coloring a graph G consists in finding an assignment of colors $c\colon V(G) \to \{1, \ldots, p\}$ such that any pair of adjacent vertices receive different colors. The minimum integer p such that a coloring exists is called the chromatic number of G, denoted by $\chi(G)$. We denote by $\Delta(G)$ the maximum degree of the graph G. Note that regarding coloring, we may only consider connected graphs. Using a greedy algorithm, it is easy to show that every graph G can be colored with $\Delta(G) + 1$ colors. A seminal result from Brooks characterizes the cases when this bound is tight:

Theorem 1 ([**2**]). *For every graph G, $\chi(G) \leq \Delta(G)$ excepted if $G = K_{\Delta(G)+1}$ or $\Delta = 2$ and G is an odd cycle.*

Given an integer $k \geq 1$ and a graph G, the k-th power of G is the graph obtained from G by adding an edge between vertices at distance at most k in G. We are interested in coloring such powers of graphs. First note that coloring powers of paths and cycles is settled by the following result, hence we may only consider graphs with maximum degree at least 3. In particular, none of the graphs we consider below is a cycle.

Proposition 2 ([**6**]). *Let n, k be two integers. Then*

- $\chi(P_n^k) = \min(n, k + 1)$

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- *If* $n > k+1$, *then* $\chi(C_n^k) = k+1+\lceil \frac{r}{q} \rceil$ *where* $n = q(k+1)+r$ *and* $r \leq k$.
- $\chi(C_n^k) = n$ *otherwise.*

For the case of squares of graphs (i.e. $k = 2$), we have $\Delta(G^2) \leq \Delta(G)^2$ (and this can be tight). Therefore, applying Brooks' theorem on G^2 states that $\Delta(G)^2$ colors are sufficient excepted when G^2 is a clique on $\Delta(G)^2 + 1$ vertices. Such graphs are called *Moore graphs*, and there are only finitely many of them [**4**]. For all the other graphs, the $\Delta(G)^2$ bound can actually be improved, as shown by the following result.

Theorem 3 ([**3**])**.** *If G is not a Moore graph, then $\chi(G^2) \leq \Delta(G)^2 - 1$.*

These results have been generalized for higher powers of graphs. Assume that $k \geq 3$. Then, the maximum possible value of $\Delta(G^k)$ is $f(k, \Delta(G))$, where

$$f(k,\Delta) = \Delta \sum_{i=0}^{k-1} (\Delta-1)^i = \Delta \frac{(\Delta-1)^k - 1}{\Delta - 2}$$

is the number of nodes of a Δ-regular tree of height k, without its root. Thus, Brooks' theorem gives that $f(k, \Delta)$ colors are sufficient to color every graph G with maximum degree Δ, as soon as it is not a generalized Moore graph, i.e. G^k is not a clique on $f(k, \Delta) + 1$ vertices. However, such a graph does not exist when $k \geq 3$ [**4**]. Therefore, the bound $\chi(G^k) \leq f(k, \Delta)$ always holds. Moreover, as shown below, this upper bound can be lowered by 1.

Theorem 4 ([**1**])**.** *For $k \geq 3$ and every graph G, we have $\chi(G^k) \leq f(k, \Delta) - 1$.*

When $k = 2$, note that $f(2, \Delta) = \Delta^2$. Hence, together with Theorem 3, this result settles a conjecture of [**5**], stating that two colors can be spared from the naive upper bound $f(k, \Delta) + 1$, excepted when $k = 2$ and G is a Moore graph.

In [**1**], the authors conjecture that we can improve this result, by sparing k colors for higher values of k, excepted for a finite number of graphs.

Conjecture 5 ([**1**])**.** For every $k \geq 2$, all but finitely many graphs G satisfy $\chi(G^k) \leq f(k, \Delta(G)) + 1 - k$.

Towards this conjecture, we prove that most of the time, $k - 2$ colors can be spared:

Theorem 6. *For every k, Δ, there are only finitely many graphs G of maximum degree Δ such that $\chi(G^k) > f(k, \Delta) + 3 - k$.*

Note that Brooks' theorem gives an infinite list of graphs such that $\chi > \Delta$. However, for every $\Delta \geq 3$, this list contains only one graph of given maximum degree Δ. In this setting, Theorem 6 can be seen as a generalization of Brooks' theorem for powers of graphs. However, observe that the bound we obtain for $k = 1$ is worse than the one given by Brooks' theorem.

2. Overview of the proof

In this section, we give a proof of Theorem 6. First note that the case $k = 1$ is easy since every graph G can be colored with $\Delta(G) + 1 \leq f(1, \Delta) + 2$. Moreover, the case $k = 2$ is already handled by Theorem 3. Thus, we only consider the case $k \geq 3$. In the following, we denote by G a graph of maximum degree $\Delta \geq 3$ such that $\chi(G^k) > f(k, \Delta) + 3 - k$, if any.

To prove Theorem 6, we prove that G cannot have some configurations, until we get to the point where G is proved not to exist at all. For each of these configurations, assuming that G contains it, we design a procedure to give a valid coloring of G, and thus reach a contradiction. This procedure roughly consists in coloring the vertices greedily by decreasing distance to the configuration. Note that if we consider a vertex far enough from the configuration, it may have at most k uncolored vertices in its neighborhood at distance k (the vertices in a shortest path to the configuration). Therefore, this kind of approach cannot conclude to a better gap than $k - O(1)$. We first apply this technique to prove that G is Δ-regular.

Lemma 7. *The graph G has minimum degree Δ.*

Proof. Assume that G has a vertex u of degree at most $\Delta - 1$. Let H be the graph obtained from G by attaching to u a pending path $u_1, \ldots, u_k$. Observe that $\Delta(H) = \Delta$ since u has degree at most $\Delta - 1$ in G. To reach a contradiction, we color vertices of G in H in order corresponding to decreasing distance to u_k.

Note that, usually, we remove elements of G, and use some minimality argument to obtain a coloring to extend. In this case, we instead add some vertices. This is not related to some inductive argument (we do not even color these new vertices). The goal of this modification is to make the gap between the number of forbidden colors and the upper bound easier to find, by counting the uncolored vertices in the neighborhood at distance k instead.

Let v be a vertex of G, at distance d from u_k. Note that the d neighbors on a shortest path from v to u_k are uncolored. Thus, v has at most $f(k, \Delta) - d$ colored neighbors in G^k, hence v has at least $d - k + 3$ available colors. Since $v \in V(G)$, we have $d \geq k$, hence v can always be colored.

We thus obtain $\chi(G^k) \leq f(k, \Delta) - k + 3$, a contradiction. □

We can prove in a similar fashion that G has large girth. Assuming that G has a small cycle, we color the vertices of G by decreasing distance to the cycle. Then, when only the cycle remains to be colored, each of its vertices has at least $p + 1$ available colors, where p is the length of the cycle. Thus we can prove the following.

Lemma 8. *The graph G has girth at least $k + 2$.*

To obtain Theorem 6, we actually prove that G has bounded diameter. Since the number of Δ-regular graphs with bounded diameter is finite, Theorem 6 thus boils down to proving the following lemma.

Lemma 9. *The graph G has diameter at most $2k-1$.*

The remainder of this section is devoted to giving the main ideas of this proof. Assume that G contains two vertices u_1 and v_k at distance $2k$ from each other. Thus there is an induced path P $u_1 \cdots u_k x v_1 \cdots v_k$ on $2k+1$ vertices in G.

For $i \in \{1, \dots, k\}$, observe that $\mathrm{dist}(u_i, v_i) = k+1$ so we can give color i to u_i and v_i. We fix the following ordering of the vertices of P: $u_1 > v_k > u_2 > v_{k-1} > \cdots > u_k > v_1 > x$. For every remaining vertex v of G, we define the *root* r_v of v as the largest vertex of P on a shortest path between x and v (in case of equality, r_v is taken as one of the v_i's).

We then color every uncolored vertex v of G by decreasing (lexicographic) order of $(r_v, \mathrm{dist}(v, r_v))$. The goal is to prove that, each time we consider a vertex v, at most $f(k, \Delta) - k + 2$ colors are present on its neighbors. To this end, we count two objects:

- The p uncolored vertices in the neighborhood of v in G^k.
- The q colors appearing at least twice in the neighborhood of v in G^k.

Note that the number of forbidden colors is then at most $f(k, \Delta) - p - q$. Thus, if $p + q \geq k - 2$, we can always find an available color for v. Lemma 9 is obtained by counting $p + q$ for each vertex v we have to color. The main idea is that if v is far away from u, then p is high. Otherwise, many neighbors of v in G^k share the same color, and q is high.

To illustrate the method, we give two examples of vertices.

- Assume that $\mathrm{dist}(v, r_v) \geq k$. Then, observe that the shortest path between v and r_v is uncolored, since every vertex on this path also has r_v as root. Therefore, v has at least $\mathrm{dist}(v, r_v) - 1$ uncolored neighbors, hence $p \geq k-2$.
- Assume that $v = x$. Then, all the colors in $\{1, \dots, k\}$ appear twice (on u_i and v_i), hence we "spare" k colors. Thus, $q \geq k$, and x can be colored.

All the other cases are treated using a combination of such arguments.

3. Conclusion and open problems

We prove that we can spare roughly k colors from the naive lower bound when coloring k-th power of many graphs. However, the only known examples that do not satisfy the bound of Conjecture 5 have small Δ. A first question is thus to know whether the bound we obtain can be strengthened using different assumptions, for example $\Delta \geq 4$, or $\Delta \geq k$.

Another natural question is about the number of exceptions given by Theorem 6. We give here a proof that all possible counterexamples of our statement have diameter less than $2k-1$, with no more insight on their structure. The coloring procedure we used to bound the diameter is quite simple, and considering more involved patterns (instead of a long path) could lead to much structure and hence a better bound.

Finally, Conjecture 5 has been stated in the list coloring setting. The arguments we use do not translate directly to this wider setting, and it would probably be interesting to determine whether Theorem 6 can be extended or not.

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