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RAMSEY NUMBERS OF BERGE-HYPERGRAPHS AND RELATED STRUCTURES

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Abstract. For a graph $G = (V, E)$, a hypergraph $\mathcal{H}$ is called a *Berge-G*, denoted by BG, if there exists an injection $f\colon E(G) \to E(\mathcal{H})$ such that for every $e \in E(G)$, $e \subseteq f(e)$. Let the Ramsey number $R^r(BG, BG)$ be the smallest integer n such that for any 2-edge-coloring of a complete r-uniform hypergraph on n vertices, there is a monochromatic Berge-G subhypergraph. In this paper, we show that the 2-color Ramsey number of Berge cliques is linear. In particular, we show that $R^3(BK_s, BK_t) = s + t - 3$ for $s, t \geq 4$ and $\max(s, t) \geq 5$ where BK_n is a Berge-K_n hypergraph. We also investigate the Ramsey number of trace hypergraphs, suspension hypergraphs and expansion hypergraphs.

1. Introduction

Given a hypergraph $\mathcal{H}$, let $v(\mathcal{H})$ denote the number of vertices of $\mathcal{H}$ and $e(\mathcal{H})$ denote the number of hyperedges. We denote the sets of vertices and hyperedges of $\mathcal{H}$ by $V(\mathcal{H})$ and $E(\mathcal{H})$, respectively. By $K_t^{(r)}$ we denote the t-vertex r-uniform clique. The set of the first n integers is sometimes denoted by $[n]$, and for a set S, we denote by $\binom{S}{r}$ the set of r-element subsets of S. Furthermore we denote the power set of a set S by 2^S. For sets A and B we denote their disjoint union by $A \sqcup B$.

Ramsey theory is among the oldest and most intensely investigated topics in combinatorics. It began with the seminal result of Ramsey from 1930.

Theorem 1 (Ramsey [**12**]). *Let r, t and k be positive integers. Then there exists an integer N such that any coloring of the N-vertex r-uniform complete hypergraph with k colors contains a monochromatic copy of the t-vertex r-uniform complete hypergraph.*

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Estimating the smallest value of such an integer N (the so-called Ramsey number) is a notoriously difficult problem and only weak bounds are known. We now give the definition of the Ramsey number for general collections of hypergraphs.

Definition 1. Let $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_k$ be nonempty collections of r-uniform hypergraphs. The Ramsey number $R^r_k(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_k)$ is defined to be the minimum integer N such that if the hyperedges of the complete r-uniform N-vertex hypergraph are colored with k colors, then for some $1 \le i \le k$, there is a monochromatic copy of a member of $\mathcal{H}_i$. If k is clear by context, then we omit k in this notation. If some of the collections $\mathcal{H}_i$ consist of a single hypergraph $\mathcal{G}$, then we write $\mathcal{G}$ in place of $\mathcal{H}_i = \{\mathcal{G}\}$.

We will primarily be concerned with families of hypergraphs defined in a natural way from a given graph G (or hypergraph $\mathcal{H}$). In the case when G is a path or a cycle, Berge [**2**] introduced a very general class of hypergraphs defined in terms of G. In particular if $G = P_t$, the path with t edges, then a Berge-P_t is any hypergraph with t hyperedges $e_1, e_2, \dots, e_t$ containing vertices $v_1, v_2, \dots, v_{t+1}$ such that $v_i, v_{i+1} \in e_i$ for all $1 \le i \le t$ (a Berge-cycle is defined analogously).

The Ramsey problem for Berge-paths and cycles has received much attention. Of particular interest is a result of Gyárfás and Sárközy [**7**] showing that the 3-color Ramsey number of a 3-uniform Berge-cycle of length n is asymptotic to $\frac{5n}{4}$.

The general definition of a Berge-G for an arbitrary graph G was introduced by Gerbner and Palmer in [**5**]. Since their publication, the Turán problem for Berge-G-free hypergraphs has been investigated heavily. Complete graphs were considered in [**10**] (and recently [**6**]). However, the analogous Ramsey problem has not yet been investigated beyond the special cases of paths and cycles.

We will recall the definition of the set of Berge-copies of a graph G. In fact, we will give a more general definition in which rather than starting with a graph G we may start with any uniform hypergraph.

Definition 2. Let $\mathcal{H} = (V, \mathcal{E})$ be a k-vertex s-uniform hypergraph. Then given an integer $r \ge s$, $B\mathcal{H}$ (the set of Berge-copies of $\mathcal{H}$) is defined to be the set of r-uniform hypergraphs $\mathcal{H}' = (W, \mathcal{F})$ such that there exist $U \subseteq W$ and bijections $\phi : V \to U$, $\psi : \mathcal{E} \to \mathcal{F}$ such that for all $e = \{u_1, u_2, \dots, u_s\} \in \mathcal{E}$, $\{\phi(u_1), \phi(u_2), \dots, \phi(u_s)\} \subseteq \psi(e)$. In this case, we call U the *core* of $\mathcal{H}'$.

One of the main topics of the present paper is determining the Ramsey number of the set of Berge-copies of a hypergraph (mainly in the graph case). We show that the 2-color Ramsey number of BK_t versus BK_s is linear. In particular, we prove the following theorem:

Theorem 2.

$$R^3(BK_s, BK_t) = \begin{cases} t+s-1 & \text{if } s=t=2,\ s=t=3 \text{ or } \{s,t\}=\{2,3\},\{2,4\}, \\ t+s-2 & \text{if } s=2, t\ge 5, \text{ or } s=3, t\ge 4 \text{ or } s=t=4, \\ t+s-3 & \text{if } s\ge 4 \text{ and } t \ge 5. \end{cases}$$

In addition to Berge-hypergraphs, we consider a variety of related structures.

Definition 3. Let $\mathcal{H} = (V, \mathcal{E})$ be a k-vertex s-uniform hypergraph and let $S \subseteq V$. The trace of $\mathcal{H}$ on S, denoted $\mathrm{Tr}(\mathcal{H}, S)$, is the hypergraph with vertex set S and hyperedge set $\{h \cap S : h \in \mathcal{E}\}$. Then, given $r \geq s$, $T\mathcal{H}$ is defined to be the set of r-uniform hypergraphs $\{\mathcal{H}' : \mathrm{Tr}(\mathcal{H}', V(\mathcal{H})) = \mathcal{H}\}$. For each such element $\mathcal{H}' \in T\mathcal{H}$, we refer to $V(\mathcal{H})$ as the core of $\mathcal{H}'$.

Definition 4. Let $\mathcal{H} = (V, \mathcal{E})$ be an s-uniform hypergraph. The r-expansion $H\mathcal{H}$, for $r \geq s$, is defined to be the r-uniform hypergraph formed by adding $r - s$ distinct new vertices to every hyperedge in $\mathcal{H}$. Precisely, for each hyperedge $e \in \mathcal{E}$, let $U_e = \{u_{e,1}, u_{e,2}, \ldots, u_{e,r-s}\}$, and define $H\mathcal{H} = (V \cup (\cup_{e \in \mathcal{E}} U_e), \mathcal{F})$ where $\mathcal{F} = \{e \cup U_e : e \in E\}$. We call V the core of $\mathcal{H}$ and $V(\mathcal{H}) \setminus V$, the set of expansion vertices.

If $\mathcal{H}$ is a cycle we recover the well-known notion of linear cycle. Ramsey and Turán problems for linear cycles have been investigated intensely (see, for example [**8**]). We investigate the 2-color Ramsey number of the 3-expansion of complete graphs K_t. By definition, a 3-expansion of complete K_t has $\binom{t}{2} + t$ vertices. Thus $R^3(HK_t, HK_t) \geq \binom{t}{2} + t$. We prove in the following theorem yielding a cubic upper bound on $R^3(HK_t, HK_s)$.

Theorem 3. *For $t, s \geq 2$, we have*

$$R^3(HK_t, HK_s) \leq 2st(s + t).$$

Next we consider another way a hypergraph can be defined from another arbitrary hypergraph called a suspension [**9**] (or earlier enlargement [**13**]).

Definition 5. Let $\mathcal{H} = (V, \mathcal{E})$ be an s-uniform hypergraph. The r-suspension $S\mathcal{H}$, for $r \geq s$, is defined to be the hypergraph formed by adding a single fixed set of $r-s$ distinct new vertices to every edge in $\mathcal{H}$. Precisely, let $U = \{u_1, u_2, \ldots, u_{r-2}\}$, and define $S\mathcal{H} = (V \cup U, \mathcal{F})$ where $\mathcal{F} = \{e \cup U : e \in E\}$. We call V the core of $S\mathcal{H}$ and U the set of suspension vertices.

For suspensions of hypergraphs, we are only able to obtain Ramsey-type bounds using standard Ramsey number techniques. In particular, we show that

Theorem 4. *For $r \geq 3$, we have*

$$(1 + o(1))\frac{\sqrt{2}}{e} t\sqrt{2}^t < R^r(SK_t, SK_t) \leq R^2(K_t, K_t) + (r - 2).$$

2. Ramsey number of Berge-hypergraphs

In this section, to avoid tedious case analysis, some of the small cases are verfied by computer. The code is available at `https://github.com/wzy3210/berge_Ramsey`.

2.1. Proof of Theorem 2

Before proving Theorem 2, we deal first with the cases when one of s or t is small. In particular, we have the following $R^3(BK_2, BK_2) = 3, R^3(BK_2, BK_3) =$

$4, R^3(BK_3, BK_3) = 5, R^3(BK_2, BK_4) = 5, R^3(BK_4, BK_4) = 6, R^3(BK_2, BK_t) = t$ when $t \geq 5$ and $R^3(BK_3, BK_t) = t+1$ when $t \geq 4$. Some cases are checked by hand, $R^3(BK_3, BK_4) = 5$ is verified by computer and $R^3(BK_3, BK_t) \leq t+1$ ($t \geq 5$) follows from Lemma 1.

Next we show the lower bound in the following proposition.

Proposition 1. *Suppose $s, t \geq 3$. We then have*

$$R^3(BK_t, BK_s) \geq t + s - 3.$$

Proof. We will construct a 2-edge-colored complete 3-uniform hypergraph $\mathcal{H}$ on $t+s-4$ vertices without blue BK_t and red BK_s. Let $V(\mathcal{H}) = A \sqcup B$ where $|A| = t-2$ and $|B| = s-2$. For all $a, a' \in A, b \in B$, color the hyperedge $\{a, a', b\}$ blue. For all $a \in A, b, b' \in B$, color the hyperedge $\{a, b, b'\}$ red. Moreover, color all triples in A blue and all triples in B red. It's easy to see that $\mathcal{H}$ is an edge-colored $K^{(3)}_{t+s-4}$ without containing blue BK_t and red BK_s. Hence $R^3(BK_t, BK_s) \geq t+s-3$. □

Before we show the proof of Theorem 2, we will prove the following lemma.

Lemma 1. *Suppose $t, s \geq 3$. Then*

$$R^3(BK_t, BK_s) \leq \max\{R^3(BK_{t-1}, BK_s), R^3(BK_t, BK_{s-1})\} + 1.$$

Proof. Let $N := \max\{R^3(BK_{t-1}, BK_s), R^3(BK_t, BK_{s-1})\} + 1$. Without loss of generality, assume $t \geq s$. Let $\mathcal{H}$ be a 2-edge-colored complete 3-uniform hypergraph with vertex set V of size at least N. We want to show that $\mathcal{H}$ contains either a blue BK_t or a red BK_s as sub-hypergraph.

Fix $v \in V$ and let $\mathcal{H}'$ be the hypergraph induced by $V' := V \setminus \{v\}$. Since $|V'| \geq R^3(BK_{t-1}, BK_s)$, it follows by definition that $\mathcal{H}'$ contains a blue BK_{t-1} or a red BK_s. If there is a red BK_s we are done. Otherwise suppose we have a blue BK_{t-1}, with the vertex set Y as its core. Now let us consider G, the blue trace of v in $\mathcal{H}$, i.e., G is a 2-edge-colored complete graph with vertex set V' and there exists an edge $\{x, y\}$ in G if and only if the hyperedge $\{x, y, v\}$ in $\mathcal{H}$ is colored blue.

Claim 1. *Either we can extend Y using v to obtain a blue BK_t or there exists a vertex $u \in Y$ with $d_G(u) \leq 1$. Moreover if $d_G(u) = 1$ and $\{u, w\}$ is the only edge containing u, then $d_G(w) < N - 2$.*

This claim says that either there exists $u \in Y$ such that $\{u, v, x\}$ is red for every $x \in V' \setminus \{u\}$, or there exists $u, w \in V'$ such that $\{u, v, x\}$ is red for every $x \neq w$ and there exists w_x such that $\{v, w, w_x\}$ is red. Note that the second case covers the first case by taking $w_x = u$. So it suffices to assume the second case.

Now since $N - 1 \geq R^3(BK_t, BK_{s-1})$, it follows that $\mathcal{H}'$ either contains a blue BK_t or a red BK_{s-1}. We are done in the former case. Otherwise, suppose that $\mathcal{H}'$ contains a red BK_{s-1}. We will show that we can extend this BK_{s-1} by adding the vertex v into its core. Let X be the core of the Berge-K_{s-1}. Now for every $x \in X$ with $x \notin \{u, w\}$, we know that the edge $\{u, v, x\}$ is colored red. Hence we can embed $\{v, x\}$ into the red hyperedge $\{u, v, x\}$. It follows that we have an embedding of the edges from v to all but at most two vertices of X, namely u, w. In the case that $w \in X$, we can embed $\{v, w\}$ into the hyperedge $\{v, w, w_x\}$, which is red. Now if $u \notin X$, we are done. Otherwise, assume $u \in X$. $|V'| = N-1 \geq \max\{R^3(BK_{t-1}, BK_s), R^3(BK_t, BK_{s-1})\} \geq s+1$. Hence it follows that there exists another vertex $y \in V(\mathcal{H}') \setminus (X \cup \{w\})$. Note that by our choice of u, $\{v, u, y\}$ is red. Thus we can embed $\{v, u\}$ into $\{v, u, y\}$. The above embedding extends X into the core of a red BK_s and we are done. □

Lemma 2. *$R^3(BK_4, BK_t) = t+1$ for $t \geq 5$.*

Proof. We will show it by induction on t. The base case that $R^3(BK_4, BK_5) = 6$ is verified by computer. The proof follows in a similar way as Lemma 1 □

Theorem 2 follows from Propositions 1 together with Lemma 1 and 2.

2.2. Superlinear lower bounds for sufficiently many colors

In this subsection we show that for all uniformities and for sufficiently many colors, the Ramsey number for a Berge t-clique is superlinear. We start with the case $r = 3$.

Claim 2. *For any $\epsilon < 1$ we have $R^3_3(BK_t, BK_t, BK_t) \geq (t-1)t^\epsilon$ for t sufficiently large.*

Proof. Let $\epsilon < 1$. Take a vertex set consisting of $t-1$ disjoint sets of vertices $V_1, V_2, \ldots, V_{t-1}$, each of size t^ϵ. If a hyperedge contains vertices from three different V_i, then color it green. By the well-known lower bound on the diagonal Ramsey number $R(K_{t^{1-\epsilon}}, K_{t^{1-\epsilon}}) = \Omega(2^{t^{1-\epsilon}/2})$, we can find a coloring of K_{t-1} containing no clique of size $t^{1-\epsilon}$ when t is sufficiently large. Given such a red-blue coloring on the complete graph with vertex set $\{1, 2, \ldots, t-1\}$ we color the hyperedges consisting of two vertices from V_i and one from V_j by the color of $\{i, j\}$ in the graph. We color every hyperedge completely contained in some V_i red. Observe that the core of any red or blue BK_t may contain vertices in less than $t^{1-\epsilon}$ different classes and so has a total of less than t vertices. □

Theorem 5. *For any uniformity $r \geq 4$, and sufficiently large c and t, we have*

$$R^r_c(BK_t, BK_t, \ldots, BK_t) > t^{1+\left(\frac{r-3}{r-2}\right)^{r-3} - \left(\frac{r-3}{r-2}\right)^{r-2}}.$$

3. Ramsey numbers of trace-cliques

Throughout the section, assume that a, b are positive integers.

Lemma 3. *$R^{a+b+1}(TK_t^{(a+1)}, TK_s^{(b+1)}) \leq R^{a+b+1}(TK_{t-1}^{(a+1)}, TK_s^{(b+1)}) + s - b$, for $t \geq a+1, s \geq b+1$.*

Theorem 6. *Let $t \geq a+1, s \geq b+1$.*
Then $R^{a+b+1}(TK_t^{(a+1)}, TK_s^{(b+1)}) \leq (t-a)(s-b) + a + b + 1$.

Proof. We are going to prove this result by induction on t, the base case is where $t = a+1$, we have that $R^{a+b+1}(TK_{a+1}^{(a+1)}, TK_s^{(b+1)}) = s+a+1 = (s-b)+b+a+1$, so the result follows. Now assume that for $t-1$ the result is true, then by lemma 3 we have that

$$R^{a+b+1}(TK_t^{(a+1)}, TK_s^{(b+1)}) \leq R^{a+b+1}(TK_{t-1}^{(a+1)}, TK_s^{(b+1)}) + s - b$$
$$\leq (t-1-a)(s-b) + a + b + 1 + (s-b) = (t-a)(s-b) + a + b + 1. \qquad \square$$

Proposition 2. *Suppose that $t \geq a+1 \geq 3$ and $s \geq 2$. Then*

$$R^{a+1}(SK_t^{(a)}, TK_s) \leq t + \max\{R^{a+1}(SK_{t-1}^{(a)}, TK_s), R^{a+1}(SK_t^{(a)}, TK_{s-1})\}.$$

Proposition 3. *Suppose that $t \geq a+1$ and $s \geq 2$. Then*

$$R^{a+1}(SK_t^{(a)}, TK_s) \geq (s-1)\left\lfloor \frac{t}{a} \right\rfloor + 1.$$

Proposition 4. *Suppose that $t \geq a+2$ and $s \geq b+2$. Then*

$$R^{a+b+1}(HK_t^{(a+1)}, TK_s^{(b+1)}) \leq M + t + b\binom{t}{a+1} - b,$$

where $M = \max\left(R^{a+b+1}(HK_{t-1}^{(a+1)}, TK_s^{(b+1)}), R^{a+b+1}(HK_t^{(a+1)}, TK_{s-1}^{(b+1)})\right)$.

4. Ramsey number of expansion and suspension hypergraphs

4.1. Expansion hypergraphs and Proof of Theorem 3

In this section, we give an upper bound on $R^3(HK_t, HK_s)$. For ease of reference, we will denote $R^r(H^r(K_t), H^r(K_t))$ by $R^3(H^3K_t, H^3K_s)$. We first show the following lemma:

Lemma 4. *For $s, t \geq 2$, we have that*

$$R^3(HK_{t+1}, HK_{s+1}) \leq \max\{R^3(HK_{t+1}, HK_s), R^3(HK_t, HK_{s+1})\} + 2st.$$

Proof. Without loss of generality, we assume that $t \leq s$. Let

$$N = \max\{R^3(HK_{t+1}, HK_s), R^3(HK_t, HK_{s+1})\} + 2st$$

and $\mathcal{H}_N$ be a 2-edge-colored compete 3-uniform hypergraph on N vertices. Let

$$W = \{v_1, v_2, \ldots, v_{2st}\} \subseteq V(\mathcal{H}_N)$$

and $\mathcal{H}' = \mathcal{H}[V(\mathcal{H}_N) \setminus W]$.

Note that $|\mathcal{H}'| \geq R^3(HK_t, HK_{s+1})$. Thus by definition of Ramsey number, there exists either a blue expansion of K_t or a red expansion of K_{s+1}. If the latter happens, we are done. Thus assume that we have a blue expansion $\mathcal{H}_b$ of K_t. Note that $\mathcal{H}_b$ has $\binom{t}{2} + t$ vertices. Let $\{u_1, \ldots u_t\}$ be the core of $\mathcal{H}_b$. Let $F = V(\mathcal{H}) \setminus V(\mathcal{H}_b)$.

Claim 3. *Suppose that $\mathcal{H}_N$ does not have a blue expansion of K_{t+1}. Then for every $v \in W$, there exists some u in the core of $\mathcal{H}_b$ such that $\{v, u, w\}$ is colored red for all w except at most $(t-1)$ elements from $F \setminus \{v\}$.*

Now since $|W| = 2st$, by pigeonhole principle, there exists some u in the core of $\mathcal{H}_b$ such that there exists $W_u = \{w_1, w_2, \dots, w_s\}$ such that for any $w \in W_u$, the hyperedge $\{w, u, w'\}$ is red for all w' except at most $(t-1)$ elements of $F \setminus \{w\}$. Let $M(w_i)$ be the elements w' in W such that $\{u, w_i, w'\}$ is blue.

Now let $W' = W_u \cup V(\mathcal{H}_b) \cup \bigcup_{i=1}^{s} M(w_i)$ and $\mathcal{H}'' = \mathcal{H}_N[V(\mathcal{H}_N) \setminus W']$. Note that $|\mathcal{H}''| \geq R^3(HK_{t+1}, HK_s)$ since $2st \geq st + \binom{t}{2} + t$. Hence there either exists a blue expansion of K_{t+1} or exists a red expansion of K_s. If the former happens, we are done. Hence assume we have a red expansion $\mathcal{H}_r$ of K_s. Suppose $\{v_1, v_2, \dots, v_s\}$ is the core of $\mathcal{H}_r$. Now we can extend $\mathcal{H}_r$ to a red expansion of K_{s+1} by adding u into the core of $\mathcal{H}_r$ together with the red edges in $\{\{u, w_i, v_i\} : i \in [s]\}$. This completes the proof of the lemma. □

4.2. Ramsey number of suspension hypergraphs

Recall that r-suspension SK_t, is the r-uniform hypergraph formed by adding a single fixed set of $r-2$ distinct new vertices to every edge in K_t. Clearly, $R^r(SK_t, SK_t) \leq R^2(K_t, K_t) + (r-2)$. The proof is simple: let $\mathcal{H}$ be a 2-edge-colored $K^{(r)}_{R^2(K_t,K_t)+(r-2)}$. Fix a set of $(r-2)$ vertices S and consider the complete graph G on the remaining $R^2(K_t, K_t)$ vertices where the color of an edge e in G is the same color as the hyperedge $e \cup S$ in $\mathcal{H}$. By standard Ramsey number, there exists a monochromatic clique in G, which gives us the core of the monochromatic SK_t in $\mathcal{H}$. The lower bound of $R^2(K_t, K_t)$ can also follow along the same line as the standard Ramsey number.

Proposition 5. *Fix $t \geq r \geq 3$. If*

$$e\left(1 + \binom{t}{2}\binom{r}{2}\binom{n}{t-2}\right) 2^{1-\binom{t}{2}} < 1,$$

then $R^r(SK_t, SK_t) > n$.

Proof. Let $\mathcal{H}$ be a complete r-uniform hypergraph on n vertices. Color each hyperedge blue/red randomly and independently with probability $\frac{1}{2}$. For a set of $r-2$ vertices S and another set of t vertices T disjoint from S, let $A_{S,T}$ be the event that the suspension hypergraph $\mathcal{H}_{S,T}$ with T as core and S as the suspending vertex set is monochromatic. Note that for each fixed S, T,

$$\Pr(A_{S,T}) = 2^{1-\binom{t}{2}} = p.$$

Note that $A_{S,T}$ is mutually independent of all other events $A_{S',T'}$ satisfying $E(\mathcal{H}_{S,T}) \cap E(\mathcal{H}_{S',T'}) = \emptyset$. Let us give an upper bound on the number of events $A_{S',T'}$ that $A_{S,T}$ is mutually dependent of. There are $\binom{t}{2}$ choices to pick an edge they share, which contains r vertices. Among the r vertices, $r-2$ of them must be the suspension vertices. There are $\binom{r}{r-2}$ ways to choose the suspension vertices S'. There are then at most $\binom{n}{t-2}$ ways to choose the remaining $t-2$ vertices of T. Hence it follows that

$$d \leq \binom{t}{2}\binom{r}{2}\binom{n}{t-2}.$$

By the Lovász Local Lemma, it follows then that if $ep(d+1) < 1$, we have that

$$\Pr\Big(\bigwedge_{S,T} \overline{A_{S,T}}\Big) > 0.$$

It follows that there exists a coloring of $\mathcal{H}$ without any monochromatic SK_t. $\square$

Remark 1. For any fixed r, this gives asymptotically the same lower bound as Ramsey number $R^2(K_t, K_t)$, i.e. $R^r(SK_t, SK_t) > (1 + o(1))\frac{\sqrt{2}}{e}t\sqrt{2}^t$.

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