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ALMOST SPANNING UNIVERSALITY IN RANDOM GRAPHS

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Abstract. A graph G is called universal for a family of graphs $\mathcal{F}$ if it contains every element $F \in \mathcal{F}$ as a subgraph. We prove for $\Delta \geq 3$ and $\varepsilon > 0$ that $G(n,p)$ is a.a.s. universal for the family of all graphs on $(1-\varepsilon)n$ vertices with maximum degree Δ provided that $p = \omega(n^{-1/(\Delta-1)})$. This improves on previously known results by Conlon, Ferber, Nenadov, and Škorić [*Almost-spanning universality in random graphs*, Random Structures Algorithms **50** (2017), 380–393] and is asymptotically optimal for $\Delta = 3$.

1. Introduction

Since the early work of Erdős and Rényi [**11**] the embedding of large structures is one of the central topics of random graph theory. After perfect matchings [**11**] and cycles [**1, 20, 24**], more recent results deal with factors [**18**] and general bounded degree graphs [**2, 8, 9, 10, 12, 13, 14, 15, 19**]. The most studied model is the *binomial random graph* $G(n,p)$, which is the model of n-vertex graphs, where each edge is present with probability p. Properties, such as subgraph containment, exhibit a threshold behaviour [**5**], which is an abrupt change for a relative small perturbation of the parameters. Formally, we call a function $\hat{p}\colon \mathbb{N} \to [0,1]$ a *threshold* for a property $\mathcal{P}$ if

$$\lim_{n\to\infty} \mathbb{P}\big[G(n,p) \in \mathcal{P}\big] = \begin{cases} 0 & \text{if } p = o(\hat{p}) \\ 1 & \text{if } p = \omega(\hat{p}). \end{cases}$$

We say that $G(n,p)$ satisfies the property $\mathcal{P}$ asymptotically almost surely (a.a.s.) if $\lim_{n\to\infty} \mathbb{P}\big[G(n,p) \in \mathcal{P}\big] = 1$

For a matching or cycle on at least $(1-\varepsilon)n$ vertices the threshold is $1/n$ for a fixed $\varepsilon > 0$, which follows from Chernoff's inequality and [**1**], respectively. At this point the expected number of perfect matchings and Hamilton cycles also gets large, but as long as $p = o(\log n/n)$ there are a.a.s. isolated vertices. It is enough to surpass this obstacle and the threshold for both is $\log n/n$ [**11, 20, 24**]. For a $K_{\Delta+1}$-factor, which are $n/(\Delta+1)$ disjoint copies of $K_{\Delta+1}$, the threshold is $(n^{-1}\log^{1/\Delta})^{2/(\Delta+1)}$, which follows from a more general result by Johannson, Kahn, and Vu [**18**]. As above, the log-term is needed to ensure that every vertex

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is contained in a copy of $K_{\Delta+1}$ and for an almost spanning $K_{\Delta+1}$-factor $n^{-2/(\Delta+1)}$ already gives the threshold, which can be proved with a standard application of Janson's inequality. Note that a K_2-factor is a perfect matching.

Turning to more general graphs, let $\mathcal{F}(n,\Delta)$ be the family of all graphs on n vertices with maximum degree Δ. We call a graph G *universal* for a family of graphs $\mathcal{F}$ if it contains any graph $F \in \mathcal{F}$ as a subgraph. Note that for large families $\mathcal{F}$ there is a difference between universality and the containment of a given $F \in \mathcal{F}$ in $G(n,p)$. First, Alon and Fredi [**2**] showed that for an integer Δ and $F \in \mathcal{F}(n,\Delta)$ a.a.s. $G(n,p)$ contains a copy of F provided that $p = \omega(\log n/n)^{1/\Delta}$. The corresponding universality result was obtained by Dellamonica, Kohayakawa, Rödl, and Ruciński [**9, 10**] for $\Delta \geq 3$ and by Kim and Lee [**19**] for $\Delta = 2$. At this probability any set of Δ vertices contains many common neighbours, which is crucial for the proof of the aforementioned results.

This natural barrier was surpassed by Conlon, Ferber, Nenadov, and kori [**8**], who proved for an $\varepsilon > 0$ that $p = \omega(n^{-1/(\Delta-1)} \log^5 n)$ gives a.a.s. $\mathcal{F}((1-\varepsilon)n,\Delta)$-universality in $G(n,p)$. Their strategy is to remove cycles until the left-over can be embedded using a result for degenerate graphs by Ferber, Nenadov, and Peter [**15**]. For $\mathcal{F}(n,\Delta)$-universality the best result is by Ferber and Nenadov [**14**] and needs $p = \omega(n^{-1}\log^3 n)^{1/(\Delta-1/2)}$. They only removed a special matching and then used the method of *robust absorption*, introduced by Montgomery [**22**], to make the embedding spanning. Ferber, Kronenberg, and Luh [**12**] obtained an optimal result for $\Delta = 2$ showing that $n^{-2/3}\log^{1/3} n$ is the threshold for $\mathcal{F}(n,2)$-universality in $G(n,p)$. We improve upon the almost spanning result, where our focus is the case $\Delta = 3$.

Theorem 1.1. *Let $\Delta \geq 3$ be an integer, $\varepsilon > 0$, and $p = \omega(n^{-1/(\Delta-1)})$. Then a.a.s. $G(n,p)$ is $\mathcal{F}((1-\varepsilon)n,\Delta)$-universal.*

This is optimal for $\Delta = 3$, because with $p = o(n^{-1/2})$ there a.a.s. is no almost spanning K_4-factor in $G(n,p)$. For $\Delta = 2$ it was already known that $n^{-2/3}$ gives the threshold [**8**]. In general it is believed that the (almost spanning) $K_{\Delta+1}$-factor is the hardest graph to embed and, therefore, $n^{-2/(\Delta+1)}$ should be the threshold for $\mathcal{F}((1-\varepsilon)n,\Delta)$-universality and $(n^{-1}\log^{1/\Delta})^{2/(\Delta+1)}$ the threshold for $\mathcal{F}(n,\Delta)$-universality. It would be very interesting to extend Theorem 1.1 to the spanning case using robust absorbers as employed in [**14**].

Our approach uses techniques of Conlon, Ferber, Nenadov, and Škorić [**8**] and Ferber and Nenadov [**14**]. A crucial ingredient of the proof is our decomposition of a graph F with maximum degree Δ (see Lemma 2.1). We will remove induced subgraphs in which the number of edges is exactly one larger than the number of vertices and prove that afterwards the graph can be sequentially dismantled. For the embedding in opposite order we further develop an embedding strategy of Ferber and Nenadov [**14**] (see Lemma 2.5 and 2.6), which, in comparison to previously known techniques, has the advantage that we do not need extra log-terms. This allows us to extend the embedding as long as there is a linear number of vertices left. To ensure universality, we work in a pseudorandom environment that satisfies the properties of $G(n,p)$ that we need (see Definition 2.2 and Proposition 2.3).

Throughout we will use standard graph theoretic notation following [**16, 17**]. In the next section we give more details of the proof.

2. Almost spanning embedding

We will now explain in more detail the ingredients for our embedding. We denote by a *chordal cycle* any connected graph with two vertices of degree 3 and all other of degree 2. These graphs are either two cycles joined by a path or one cycle with a path connecting two vertices. The length of a chordal cycle is the number of its vertices.

Lemma 2.1. *For any integer $\Delta \geq 3$ and $F \in \mathcal{F}(n,\Delta)$ there exist integers $t_1 \leq t_2 \leq t$ and a sequence of graphs $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_t = F$ such that F_0 is the empty graph and the following holds:*

(i) *If $0 < i \leq t_1$, then $F_i \setminus F_{i-1}$ is a K_k with $2 \leq k \leq \Delta+1$, which is isolated in F_{i-1}.*

(ii) *If $t_1 < i \leq t_2$, then $F_i \setminus F_{i-1}$ is a single vertex or an edge, where each vertex has at most one neighbour in F_{i-1}.*

(iii) *If $t_2 < i \leq t$, then $F_i \setminus F_{i-1}$ is a chordal cycle of length at most $16\log n$.*

This decomposition is inspired by a lemma from Conlon, Ferber, Nenadov, and Škorić [**8**] for finding a cycle in the $\log n$ neighbourhood of a vertex and by a lemma from Krivelevich [**21**], which implies that any tree either has many leaves or many bare paths. For embedding all graphs with this decomposition we define the following pseudorandom properties.

Definition 2.2. For $\Delta \geq 3$, $\eta > 0$, and $p \in (0,1)$ we say that an n-vertex graph G is an (n,p,η,Δ)-*graph* if there is a partition $\mathcal{U} = \{U_0,\dots,U_{\Delta+1}\}$ of $V(G)$ with $|U_i| = \eta n$ for $1 \leq i \leq \Delta+1$ such that the following holds:

(A1) For any $V \subseteq V(G)$ with $|V| \geq \eta n$ there is a copy of $K_{\Delta+1}$ in $G[V]$.

(A2) For any $\mathcal{L} \subseteq V(G)$ of size $|\mathcal{L}| = \ell$, $U \in \mathcal{U}$, and $U' \subseteq U$ a subset of size $|U'| \geq \max\{\eta|U|, |U| - \ell n^{1/2}/\log n\}$ there exists vertices $u \in \mathcal{L}$ and $v \in U'$ such that vu is an edge in G.

(A3) For any $\mathcal{L} \subseteq V(G)^2$ a set of disjoint tuples of size $|\mathcal{L}| = \ell$, $U \in \mathcal{U}$, and $U' \subseteq U$ a subset of size $|U'| \geq \max\{\eta|U|, |U| - \ell n^{1/2}/\log n\}$ there exists a pair $(u_1,u_2) \in \mathcal{L}$ and and an edge $v_1v_2 \in G[U']$ such that v_1u_1 and v_2u_2 are edges in G.

(A4) For any chordal cycle F on vertices $v_1,\dots,v_k$ with $4 \leq k \leq 16\log n$, $\mathcal{L} \subseteq V(G)^{(\Delta-2)k-2}$ a set of disjoint $((\Delta-2)k-2)$-tuples $(u_1,\dots,u_{(\Delta-2)k-2})$ of size $|\mathcal{L}| = \ell$, $U \in \mathcal{U}$, and $U' \subseteq U$ a subset of size $|U'| \geq \max\{\eta|U|, |U| - \ell n^{1/(\Delta-1)}/\log n\}$ there exists an $(u_1,\dots,u_{(\Delta-2)k-2}) \in \mathcal{L}$ and a copy of F in G with each v_i mapped to $\tilde{v}_i \in U'$ for $1 \leq i \leq k$ such that $\tilde{v}_i u_{(\Delta-2)(i-1)+j}$ is an edge in G for $1 \leq j \leq \Delta-2$ when $1 \leq i \leq k-2$, $1 \leq j \leq \Delta-3$ when $i = k-1$, and $0 \leq j \leq \Delta-4$ when $i = k$.

Property **(A1)** allows us us to embed cliques into prescribed vertexsets. The others, **(A2)**–**(A4)**, enable us to embed one graph with connections from a large enough family of graphs. For example, for any family $\mathcal{L}$ of $n^{(\Delta-2)/(\Delta-1)}\log n$ many vertices in G and any set $U' \subseteq U_0$ of size $|U'| \geq \eta n$ by (**A2**) one vertex from U' is incident to a vertex from $\mathcal{L}$. The special sets $U_1, \dots, U_{\Delta+1}$ will be used, when we already embedded a lot into U_0 and $\mathcal{L}$ is smaller. We denote the family of (n,p,η,Δ)-graphs by $\mathcal{G}(n,p,\eta,\Delta)$ and show that $G(n,p)$ a.a.s. is in $\mathcal{G}(n,p,\eta,\Delta)$.

Proposition 2.3. *For $\Delta \geq 3$, $1/(\Delta+2) \geq \eta > 0$, and $p = \omega(n^{-1/(\Delta-1)})$ the random graph $G(n,p)$ a.a.s. is in $\mathcal{G}(n,p,\eta,\Delta)$.*

Finding a copy of $K_{\Delta+1}$ in any linear sized set can easily be proved using Janson's inequality (see [**17**, Theorem 2.18]), which implies (**A1**). Property (**A2**) follows by a simple Chernoff bound (see [**17**, Theorem 2.8]). For (**A3**) and (**A4**) we can again use Janson's inequality and similar calculations to Conlon, Ferber, Nenadov, and kori [**8**]. It then remains to prove the following deterministic embedding statement.

Theorem 2.4. *For $\Delta \geq 3$, $\varepsilon > 0$, $\varepsilon/(\Delta+2) \geq \eta > 0$, and $p = \omega(n^{-1/(\Delta-1)})$ let $G \in \mathcal{G}(n,p,\eta,\Delta)$. Then G is $\mathcal{F}((1-\varepsilon)n,\Delta)$-universal.*

Together with Proposition 2.3 this immediately implies Theorem 1.1. This will in turn be implied by our decomposition result, Lemma 2.1, together with (**A1**) for embedding initial K_k's with $2 \leq k \leq \Delta+1$ and the following two lemmas, which use (**A2**), (**A3**), and (**A4**) to embed the rest.

Lemma 2.5. *For $\Delta \geq 3$, $\varepsilon > 0$, $\varepsilon/(\Delta+2) \geq \eta > 0$, and $p = \omega(n^{-1/2})$ let $G \in \mathcal{G}(n,p,\eta,\Delta)$ with $\mathcal{U} = \{U_0,\dots,U_{\Delta+1}\}$ and $G' = G - (U_1 \cup \dots \cup U_{\Delta-1})$. Further, let F be any graph on at most $(1-\varepsilon)n$ vertices and $S \subset V(F)$ such that there exists a sequence $F[S] = F_0 \subseteq \dots \subseteq F_t = F$ such that for $0 < i \leq t$ the graph $F_i \smallsetminus F_{i-1}$ is a single vertex or edge, where each vertex has at most one neighbour in F_{i-1}. Then any embedding of F_0 into $G[U_0]$ can be extended to an embedding of F into G'.*

Lemma 2.6. *For $\Delta \geq 3$, $\varepsilon > 0$, $\varepsilon/(\Delta+2) \geq \eta > 0$, and $p = \omega(n^{-1/(\Delta-1)})$ let $G \in \mathcal{G}(n,p,\eta,\Delta)$ with $\mathcal{U} = \{U_0,\dots,U_{\Delta+1}\}$. Further, let $F \in \mathcal{F}((1-\varepsilon)n,\Delta)$ and $S \subset V(F)$ such that there exists a sequence $F[S] = F_0 \subseteq \dots \subseteq F_t = F$ such that for $0 < i \leq t$ the graph $F_i \smallsetminus F_{i-1}$ is a chordal cycle of length at most $16\log n$. Then any embedding of F_0 into $G[U_0 \cup U_\Delta \cup U_{\Delta+1}]$ can be extended to an embedding of F into G.*

To prove these lemmas we use a strategy of Ferber and Nenadov [**14**]. The main idea is to embed the new vertices always into the U_i with i as small as possible. Then with (**A2**)–(**A4**) we can show that most vertices are embedded into U_0, less and less are embedded into the U_i for larger i, and, most importantly, the embedding is successful for all vertices. With this at hand it is easy to prove Theorem 2.4.

Proof of Theorem 2.4. Let $\Delta \geq 3$, $\varepsilon > 0$, $\varepsilon/(\Delta+2) \geq \eta > 0$, $p = \omega(n^{-1/(\Delta-1)})$, and $G \in \mathcal{G}(n,p,\eta,\Delta)$. Then for any $F \in \mathcal{F}((1-\varepsilon)n,\Delta)$ we apply Lemma 2.1. We repeatedly use (**A1**) to obtain an embedding of F_{t_1} into $G[U_0]$. Then with

Lemma 2.5 we extend this to an embedding of F_{t_2} into G avoiding $U_1, \ldots, U_{\Delta-1}$. Finally, we can use Lemma 2.6 to finish the embedding of F. □

3. Concluding remarks

Almost spanning embeddings into random graphs are very helpful for proving results in the model of *randomly perturbed graphs*. This model is at the intersection of random and extremal graph theory and was introduced by Bohmann, Frieze, and Martin [**4**]. For an $\alpha > 0$ it is the union of any graph G_α with minimum degree αn and $G(n,p)$. In $G_\alpha \cup G(n,p)$ we do not need extra log-terms to guarantee a certain minimum degree and $p = \omega(1/n)$ suffices a.a.s. for a Hamilton cycle [**4**]. Similarly, for the $K_{\Delta+1}$-factor [**3**] and also for embedding one $F \in \mathcal{F}(n,\Delta)$ [**7**] the probability $p = \omega(n^{1/(2\Delta+1)})$ a.a.s. is enough in $G_\alpha \cup G(n,p)$. In general we expect that in $G_\alpha \cup G(n,p)$ the threshold for the almost spanning results in $G(n,p)$ are sufficient, while most of them are optimal because G_α can be $K_{\alpha n,(1-\alpha)n}$. For universality only trees [**6**] and $\mathcal{F}(n,2)$ [**23**] were considered using the approach from [**7**]. Following [**6, 23**] we can use our approach for almost spanning universality in $G(n,p)$ to obtain the following. For every $\alpha > 0$ we have with $p = \omega(n^{-1/(\Delta-1)})$ that $G_\alpha \cup G(n,p)$ is $\mathcal{F}(n,\Delta)$-universal. This is optimal for $\Delta = 3$, while for larger Δ the conjecture is $n^{2/(\Delta+1)}$.

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