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THE MAXIMUM NUMBER OF P_ℓ COPIES IN P_k-FREE GRAPHS

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Abstract. Generalizing Turán's classical extremal problem, Alon and Shikhelman investigated the problem of maximizing the number of copies of T in an H-free graph, for a pair of graphs T and H. Whereas Alon and Shikhelman were primarily interested in determining the order of magnitude for some classes of graphs H, we focus on the case when T and H are paths, where we find asymptotic and exact results in some cases. We also consider other structures like stars and the set of cycles of length at least k, where we derive asymptotically sharp estimates. Our results generalize well-known extremal theorems of Erdős and Gallai.

1. Introduction

For a graph G, we let $e(G)$ denote the number of edges in G, and for a given graph H, we let $\mathcal{N}(H, G)$ denote the number of (not necessarily induced) copies of H in G. If there is no copy of H in G, we say that G is H-free. We denote the path with k edges by P_k and the cycle with k edges by C_k. By $C_{\geq k}$ we mean the set of all cycles of length at least k. By S_k we denote the star on $k+1$ vertices. Given a graph G containing a vertex v, we denote the neighborhood of v by $N(v)$. The independence number, minimum degree and number of vertices in G are denoted by $\alpha(G)$, $\delta(G)$ and $v(G)$, respectively. The vertex and edge sets of G are denoted by $V(G)$ and $E(G)$, respectively. Finally, given a set $S \subseteq V(G)$, we denote by $G[S]$ the induced subgraph of G with vertex set S.

Following the notation of Alon and Shikhelman [**2**], we let $\mathrm{ex}(n, T, H)$ be the maximum number of (noninduced) copies of T in an H-free graph on n vertices. Observe that we have $\mathrm{ex}(n, P_1, H) = \mathrm{ex}(n, H)$, the classical extremal number. If a set of graphs $\mathcal{H}$ is forbidden, then we define $\mathrm{ex}(n, \mathcal{H})$ (and similarly $\mathrm{ex}(n, T, \mathcal{H})$), in the obvious way.

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We begin by recalling the famous theorem of Erdős and Gallai on P_k-free graphs as well as some recent generalizations due to Luo, where the number of cliques is considered.

Theorem 1 (Erdős–Gallai [**6**]). *For all $n \geq k$,*

$$\mathrm{ex}(n, P_k) \leq \frac{(k-1)n}{2},$$

and equality holds if and only if k divides n and G is the disjoint union of cliques of size k.

In their paper, Erdős and Gallai deduced Theorem 1 as a corollary of the following result about graphs with no long cycles.

Theorem 2 (Erdős–Gallai [**6**]). *For all $n \geq k$,*

$$\mathrm{ex}(n, C_{\geq k}) \leq \frac{(k-1)(n-1)}{2},$$

and equality holds if and only if $k-2$ divides $n-1$ and G is a connected graph such that every block of G is a clique of size $k-1$.

As the extremal examples for Theorem 1 are disconnected, it is natural to consider a version of the problem where the base graph is assumed to be connected. Kopylov [**10**] settled this problem, and later Ballister, Győri, Lehel and Schelp [**3**] classified the extremal cases.

Definition 1. We denote by $G_{n,k,a}$ the graph whose vertex set is partitioned into 3 classes, A, B and C with $|A| = a$, $|B| = n-k+a$, $|C| = k-2a$ such that $A \cup C$ induces a clique, B is an independent set and all possible edges are taken between vertices of A and B.

Throughout this paper we let $t = \left\lfloor \frac{k-1}{2} \right\rfloor$. In $G_{n,k,t}$, the class C has one vertex when k is odd or two vertices when k is even. By grouping B and C together, we have that $G_{n,k,t}$ is obtained from a complete bipartite graph $K_{t,n-t}$ by adding all edges in the color class of size t, and in the case that k is even, adding one additional edge inside the color class of size $n-t$.

Theorem 3 (Kopylov [**10**], Ballister–Győri–Lehel–Schelp [**3**]). *Let G be a connected n-vertex P_k-free graph, with $n \geq k$, then*

$$e(G) \leq \max(e(G_{n,k,t}), e(G_{n,k,1})).$$

Moreover, the extremal graph is either $G_{n,k,t}$ or $G_{n,k,1}$.

Note that if $n \geq 5k/4$, this maximum is achieved by $G_{n,k,t}$. Also observe that

$$e(G_{n,k,t}) = t(n-t) + \binom{t}{2} + \eta_k,$$

where η_k is 1, if k is even, and 0 otherwise. Thus, Theorems 1 and 3 yield the same bound asymptotically as n tends to infinity.

The following theorem was deduced by Luo [**12**] as a corollary of her main result but also follows from Theorem 1 using a simple induction argument. We present this proof here.

Theorem 4 (Luo [**12**]).

$$\mathrm{ex}(n, K_r, P_k) \le \frac{n}{k}\binom{k}{r}.$$

Proof. We use induction on r, and the base is Theorem 1. Let G be an n-vertex graph containing no P_k. We have

$$\begin{aligned}\mathcal{N}(K_r, G) &= \frac{1}{r}\sum_{v\in V(G)} \mathcal{N}(K_{r-1}, G[N(v)]) \\ &\le \frac{1}{r}\sum_{v\in V(G)} \frac{v(G[N(v)])}{k-1}\binom{k-1}{r-1} \\ &= \frac{1}{k(k-1)}\binom{k}{r}2e(G),\end{aligned}$$

since $G[N(v)]$ contains no P_{k-1}. By Theorem 1, we have $e(G) \le \frac{(k-1)n}{2}$, and the result follows. □

For our results we will need only that $\mathrm{ex}(n, K_r, P_k) \le c_{k,r}n$ for some constant $c_{k,r}$ depending only on k and r.

If we impose the additional condition that the graph is connected, then the situation is more complicated. Luo proved the following sharp bounds.

Theorem 5 (Luo [**12**]). *Let $n > k \ge 3$ and G be a connected n-vertex graph with no path of length k, then*

$$\mathcal{N}(K_r, G) \le \max\left(\mathcal{N}(K_r, G_{n,k,t}), \mathcal{N}(K_r, G_{n,k,1})\right).$$

Theorem 6 (Luo [**12**]). *Let $n \ge k \ge 4$ and G be a n-vertex graph with no cycle of length k or greater, then*

$$\mathcal{N}(K_r, G) \le \frac{n-1}{k-2}\binom{k-1}{r}.$$

Some recent generalizations of the Erdős–Gallai theorem and Luo's results can be found in [**14**]. In the present paper we focus on results where paths or all sufficiently long cycles are forbidden. The general problem of enumerating cycles of a fixed length when a fixed cycle is forbidden has also been considered recently (see [**7**] and [**8**] which generalize earlier results for special cases, e.g., [**4**], [**9**], [**2**]).

Alon and Shikhelman [**2**] considered the problem of maximizing the number of copies of a tree T in a graph which is H-free, for another tree H. Given two trees T and H, they introduced an integer parameter $m(T, H)$ and proved that $\mathrm{ex}(n, T, H) = \Theta(n^{m(T,H)})$, thereby determining the correct order of magnitude for all pairs of trees. A recent result due to Letzter [**11**] extends the above result of Alon and Shikhelman to the case when only H is a tree and T is arbitrary. It is shown that, nonetheless, the order of magnitude of $\mathrm{ex}(n, T, H)$ is a positive integer power of n.

In the present paper, we are interested in the case where the forbidden tree is a path, and we find correct asymptotics and sometimes the exact bound for the maximum number of copies of a smaller path (as well as for several other types of graphs). We also obtain asymptotic results for the problem of maximizing copies of T in a graph with no cycles of length at least k, in the case when T is a path.

2. Asymptotic Results

We write $f(n,k) \sim g(n,k)$ when $\lim_{k\to\infty}\left(\lim_{n\to\infty}\frac{f(n,k)}{g(n,k)}\right) = 1$. We estimate on the number of copies of paths and cycles in a P_k-free graph. For a fixed natural number ℓ, we prove the following asymptotic results:

Theorem 7.

$$\mathrm{ex}(n, P_{2\ell}, P_k) \sim \frac{k^\ell n^{\ell+1}}{2^{\ell+1}}.$$

Theorem 8.

$$\mathrm{ex}(n, P_{2\ell+1}, P_k) \sim \frac{(\ell+2)k^{\ell+1}n^{\ell+1}}{2^{\ell+2}}.$$

Theorem 9.

$$\mathrm{ex}(n, C_{2\ell}, P_k) \sim \frac{k^\ell n^\ell}{\ell 2^{\ell+1}}.$$

Theorem 10.

$$\mathrm{ex}(n, C_{2\ell+1}, P_k) \sim \frac{k^{\ell+1}n^\ell}{2^{\ell+2}}.$$

The construction showing the lower bounds for Theorems 7 through 10 is the same as the extremal construction for the connected version of the Erdős–Gallai theorem, Theorem 3. Because we are interested in asymptotics, we will omit one edge from this construction which only occurs when k is even. Our n-vertex graph G is defined by taking a clique on a set S of $\left\lfloor \frac{k-1}{2} \right\rfloor$ vertices and connecting every vertex in S to every vertex of an independent set U, defined on $n - \left\lfloor \frac{k-1}{2} \right\rfloor$ vertices. It is easy to see that this graph is P_k-free. In enumerating the copies of $P_{2\ell}$, the only paths which contribute asymptotically alternate between S and U, starting and ending with U (the factor of 2 comes from counting the path in both directions).

When enumerating the copies of $P_{2\ell+1}$, we have two kinds of paths which contribute asymptotically: those that start and end in U, using an edge in S at some step, and those that start in U and end in S, never using an edge contained in S. For the first type, we condition on which step in the path we use the edge in S (ℓ possibilities). Each such path gets counted twice, hence we divide by two. For the second type, each path is counted once and so we do not have to divide by 2.

We begin by showing how Theorem 7 can be derived from a result about the spectral radius of P_k-free graphs due to Nikiforov [**13**]. Recall that the spectral radius of a graph G is the maximum of the eigenvalues of the adjacency matrix of G. He determined, for sufficiently large n, the maximal spectral radius of a P_k-free graph on n vertices. We are interested in asymptotics so we will make use of the following corollary which follows directly from the results in [**13**].

Corollary 1 (Nikiforov)**.** *If n is sufficiently large and G is a P_k-free graph, then the spectral radius of G is at most $\sqrt{\lfloor (k+1)/2 \rfloor n}$.*

Spectral proof of Theorem 7. Let G be a P_k-free graph on n vertices (for n large enough to satisfy Corollary 1). Let A be the adjacency matrix of G, then we have

$$\begin{aligned} 2 \cdot \frac{\mathcal{N}(P_{2\ell}, G)}{n} &\leq \frac{\#2\ell\text{-walks in } G}{n} \\ &= \frac{\mathbf{1}^t A^{2\ell} \mathbf{1}}{\mathbf{1}^t \mathbf{1}} \\ &\leq \left(\sqrt{\lfloor (k+1)/2 \rfloor n} \right)^{2\ell} \\ &= (\lfloor (k+1)/2 \rfloor n)^{\ell}. \end{aligned}$$

Where $\mathbf{1}$ is the all 1's vector, and the second inequality comes from the fact that the spectral radius of a Hermitian matrix M is the supremum of the quotient $\frac{x^* M x}{x^* x}$. Therefore, for every natural number k and n sufficiently large we have $\mathrm{ex}(n, P_{2\ell}, k) \leq n^{\ell+1} \lfloor (k+1)/2 \rfloor^{\ell}/2$. $\square$

Unfortunately, it does not seem like this approach can be used to prove Theorem 8 as the bound it would yield is off by a factor of order $\sqrt{n}$.

References

1. Alon N., *On the number of subgraphs of prescribed type of graphs with a given number of edges*, Israel J. Math. **38** (1981), 116–130.
2. Alon N. and Shikhelman C., *Many T copies in H-free graphs*, J. Combin. Theory Ser. B **121** (2016), 146–172.
3. Balister P., Győri E., Lehel J. and Schelp R., *Connected graphs without long paths*, Discrete Math. **308** (2008), 4487–4494.
4. Bollobás B. and Győri E., *Pentagons vs. triangles*, Discrete Math. **308** (2008), 4332–4336.
5. Dirac G., *Some theorems on abstract graphs* Proc. Lond. Math. Soc. (3) **2** (1952), 69–81.
6. Erdős P. and Gallai T., *On maximal paths and circuits of graphs*, Acta Math. Hungar. **10** (1959), 337–356.
7. Gishboliner L. and Shapira A., *A generalized Turán problem and its applications*, `arXiv:1712.00831`.
8. Gerbner D., Győri E., Methuku A. and Vizer M., *Generalized Turán problems for even cycles*, `arXiv:1712.07079`.
9. Győri E. and Li H., *The maximum number of triangles in C_{2k+1}-free graphs*, Combin. Probab. Comput. **21** (2012), 187.
10. Kopylov G., *Maximal paths and cycles in a graph*, Dokl. Akad. Nauk **234** (1977), 19–21.
11. Letzter S., *Many H-copies in graphs with a forbidden tree*, `arXiv:1811.04287`.
12. Luo R., *The maximum number of cliques in graphs without long cycles*, J. Combin. Theory Ser. B **128** (2018), 219–226.
13. Nikiforov T., *The spectral radius of graphs without paths and cycles of specified length*, Linear Algebra Appl. **432** (2010), 2243–2256.
14. Ning B. and Peng X., *Extensions of Erdős–Gallai theorem and Luo's theorem with applications*, `arXiv:1801.09981`.

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