Acta Math. Univ. Comenianae
Vol. LXXXVIII, 3 (2019), pp. 1017–1021

THE KUPERBERG CONJECTURE FOR TRANSLATES OF CONVEX BODIES

R. PROSANOV

Abstract. We prove that if a convex body C admits a dense translative packing, then it admits an economical translative covering and vice versa. This answers positively to the question of W. Kuperberg in the case of translative arrangements.

1. Definitions and statements of results

Let C be a d-dimensional convex body in $\mathbb{R}^d$, i.e. a compact convex set with nonempty interior. An *arrangement* $\mathcal{C} = \{C_1, C_2, \ldots\}$ is a countable collection of congruent copies of C in $\mathbb{R}^d$. An arrangement is called *packing* if no two distinct copies in $\mathcal{C}$ have an interior point in common. An arrangement is called *covering* if $\mathbb{R}^d = \bigcup_i C_i$.

Define *upper* and *lower* densities of an arrangement

$$\overline{\mathrm{den}}(\mathcal{C}) = \limsup_{r\to\infty} \frac{\sum_{C_i\in\mathcal{C}} \mathrm{vol}\big(C_i \cap B^d(r)\big)}{\mathrm{vol}(B^d(r))},$$

$$\underline{\mathrm{den}}(\mathcal{C}) = \liminf_{r\to\infty} \frac{\sum_{C_i\in\mathcal{C}} \mathrm{vol}\big(C_i \cap B^d(r)\big)}{\mathrm{vol}(B^d(r))},$$

where $B^d(r)$ is the Euclidean ball of radius r centered at the origin.

The *packing density* of C is

$$\delta(C) = \sup_{\mathcal{C} \text{ is a packing}} \overline{\mathrm{den}}(\mathcal{C}). \tag{1}$$

Similarly, the *covering density* of C is

$$\theta(C) = \inf_{\mathcal{C} \text{ is a covering}} \underline{\mathrm{den}}(\mathcal{C}). \tag{2}$$

Note that these densities are reached. It is also clear that $\delta(C) \leq 1$ and $\theta(C) \geq 1$. Then using averaging and compactness arguments one can show

$$\delta(C) = 1 \iff \theta(C) = 1 \iff C \text{ is a tile.}$$

Received June 5, 2019.
2010 *Mathematics Subject Classification.* Primary 52C17, 05B40, 52A23.
Supported by the grant from Russian Federation President Programme of Support for Leading Scientific Schools 6760.2018.1 and SNF grant 200021_−169391 "Discrete curvature and rigidity".

This was proved by Schmidt [**13**], one can also look in [**5**, p. 805].

It is a natural question to ask if this property is stable. In other words, if a body C cannot be packed densely, does it mean that it cannot cover $\mathbb{R}^d$ economically? This was initially conjectured by W. Kuperberg [**1**, Chapter 1.10, Conjecture 1]:

Conjecture 1.1. Let $d \geq 2$ be fixed. Then for any $\varepsilon > 0$ there exists $\delta > 0$ with the property that for every d-dimensional convex body C

$$(1) \qquad \theta(C) \geq 1 + \varepsilon \qquad \text{implies} \qquad \delta(C) \leq 1 - \delta,$$

$$(2) \qquad \delta(C) \leq 1 - \varepsilon \qquad \text{implies} \qquad \theta(C) \geq 1 + \delta.$$

In fact, G. Fejes-Tóth and W. Kuperberg ([**4**], [**10**]) asked to investigate the links between packing and covering densities in more details. Let $\Omega_d \subset \mathbb{R}^2$ be the set of points (x, y) such that there exists a d-dimensional convex body C with the property $\delta(C) = x$, $\theta(C) = y$. It is a natural question to describe this set explicitly. Unfortunately, even in the case $d = 2$ this problem seems to be very hard. For instance, it is not known whether this set is closed or simply-connected. It is conjectured that Ω_d is convex.

There are some restrictions that are frequently imposed on the arrangements $\mathcal{C}$ under consideration. It is interesting to restrict our attention to the case of translative or lattice arrangements. An arrangement $\mathcal{C} = \{C_1, C_2, \ldots\}$ is called *translative* if all C_i are translates of the body C. It is called a *lattice arrangement* if in addition the translation vectors form a lattice Λ.

For a convex body C we define its translative packing and covering densities $\delta_T(C)$ and $\theta_T(C)$ by considering the supremum or the infimum in the expressions (1) or (2) respectively with the additional condition that the arrangements are translative. Similarly, we denote lattice packing and covering densities by $\delta_L(C)$ and $\theta_L(C)$.

Our main result provides a link between translative packing and covering densities in all dimensions:

Theorem 1.2. *Let $d \geq 2$ and C be a d-dimensional convex body.*

(1a) *Let either $0 < \varepsilon \leq \frac{1}{d^{d+1}}$ or C in addition be centrally symmetric. If for the translative packing density we have $\delta_T(C) > 1 - \varepsilon$, then the translative covering density of C satisfies*

$$\theta_T(C) < \left(1 + \varepsilon^{\frac{1}{d+1}}\right)^{d+1}.$$

(1b) *Let $\frac{1}{d^{d+1}} < \varepsilon < 1$ and C be not centrally symmetric. If for the translative packing density we have $\delta_T(C) > 1 - \varepsilon$, then the translative covering density of C satisfies*

$$\theta_T(C) < \left(1 + \varepsilon d^d\right)\left(1 + \frac{1}{d}\right)^d.$$

(2) *Let $0 < \varepsilon < 1$. If for the translative covering density we have $\theta_T(C) < 1 + \varepsilon$, then the translative packing density of C satisfies*

$$\delta_T(C) > \left(1 - \varepsilon^{\frac{1}{d+1}}\right)^{d+1}.$$

In particular, this establishes Conjecture 1.1 in the case of translative densities. Indeed, in Conjecture 1.1 it is enough to consider only sufficiently small values of ε. If $\varepsilon \le \left(1+\frac{1}{d}\right)^{d+1} - 1$ (to satisfy the condition in part (1a) of Theorem 1.2), then one can take

$$\delta = \left((1+\varepsilon)^{\frac{1}{d+1}} - 1\right)^{d+1}$$

and obtain (1) in Conjecture 1.1 from (1a) in Theorem 1.2. If $\varepsilon < 1$, then take

$$\delta = \left(1 - (1-\varepsilon)^{\frac{1}{d+1}}\right)^{d+1}$$

and get (2) in Conjecture 1.1 from (2) in Theorem 1.2.

2. Comparison with previously known results

The only known case of Conjecture 1.1 was proven by D. Ismailescu for $d = 2$, with the additional conditions that C is centrally symmetric and arrangements are lattice or translative. More precisely, in [**7**] he showed that under these assumptions

$$1 - \delta_L(C) \le \theta_L(C) - 1 \le 1.25\sqrt{1 - \delta_L(C)}. \tag{3}$$

For centrally symmetric planar bodies it was already proven by L. Fejes-Tóth [**6**] that $\delta_L(C) = \delta_T(C)$ and by C. Dowker [**2**] that $\theta_L(C) = \theta_T(C)$. Thereby, this implies Conjecture 1.1 in these cases. The proof of (3) is based on the approximation of C by centrally symmetric octagons and cannot be extended to higher dimensions. We also remark that it is widely believed that in higher dimensions $\delta_L(C) \ne \delta_T(C)$ and $\theta_L(C) \ne \theta_T(C)$.

Several other results linking together packing and covering densities of convex bodies are given in [**8**], [**9**] and [**15**]. All of them are known to hold only in dimensions 2 or 3.

It is interesting to compare under which condition on $\delta_T(C)$, Theorem 1.2 gives us a better bound on $\theta_T(C)$ than a general best known bound for covering densities. The similar comparison can be provided with general bounds on packing densities. For the sake of brevity, we give the conclusion only in the case when C is centrally symmetric.

The best known bound for $\theta_T(C)$ is due to G. Fejes-Tóth [**3**]:

$$\theta_T(C) \le d\ln d + d\ln\ln d + d + o(d). \tag{4}$$

In the centrally symmetric case Theorem 1.2 gives us a better bound than (4) provided that

$$1 - \delta_T(C) < \left(\frac{\ln(d\ln d + d\ln\ln d + d)}{d+1}\right)^{d+1}.$$

For the packing densities of centrally symmetric bodies the best result was obtained by W. Schmidt [**14**]:

$$\delta_T(C) \ge \delta_L(C) \ge \frac{d\ln\sqrt{2}}{2^d} - o(1), \tag{5}$$

Theorem 1.2 gives us a better bound than (5) if

$$\theta_T(C) - 1 < \left(\frac{1}{2} - \frac{\ln(d\ln 2)}{d+1}\right)^{d+1}.$$

3. Sketch of proof

We call a translative arrangement $\mathcal{C}$ of translates of C to be *periodic* if the translation vectors form a set of the form $\Lambda + X$, where Λ is a d-dimensional lattice in $\mathbb{R}^d$ and X is a finite point set. The following theorem of Rogers [**12**, Theorems 1.7 and 1.9] allows us in the framework of our problem to reduce the study of translative arrangements to the study of periodic ones:

Theorem 3.1. *For a convex body C in the definition of its translative packing density we can take the supremum over only periodic arrangements. The same holds for its translative covering density.*

Now we sketch the proof of the first part of Theorem 1.2. For the details we refer the reader to the extended paper [**11**].

Consider a periodic packing $\mathcal{C}$ of the form $C + \Lambda + X$ of the density $\mathrm{den}(\mathcal{C}) > \delta_T(C) - \varepsilon$. We proceed to the torus $T = \mathbb{R}^d/\Lambda$ and abusing the notation we still denote the images of C and X under the projection map $\mathbb{R}^d \to T$ by C and X. Then we take α: $0 < \alpha < 1$ and consider the arrangement $(1+\alpha)C + X$ on T. If it is not a covering of T, then we choose a point $y_1 \in T$, which is not covered. It is easy to see that the interior of $-\alpha C + y_1$ does not intersect the interiors of $C + X$. We take $y_1' \in T$ such that $C + y_1'$ covers $-\alpha C + y_1$. If C is centrally symmetric, then we can set $y_1' = y_1$, in the other case such a point exists by a result of Minkowski and Radon under the additional assumption that $\alpha \le \frac{1}{d}$ (see e.g. [**16**, Corollary 1.4.2]).

Now we consider $X_1 = X \cup \{y_1'\}$ and repeat our operation obtaining points $y_2, y_2' \in T$ and the set $X_2 = X_1 \cup \{y_2'\}$. We continue this process until for some l we obtain that $(1+\alpha)C + X_l$ covers T.

Then we get a bound

$$\begin{aligned}\theta_T(C) &= \theta_T((1+\alpha)\,C) \le \mathrm{den}\left((1+\alpha)\,C + X_l + \Lambda\right)\\ &= \frac{|X_l|\,(1+\alpha)^d\,\mathrm{vol}\,(C)}{\mathrm{vol}(T)} = (1+\alpha)^d\left(\mathrm{den}(\mathcal{C}) + \frac{l\mathrm{vol}\,(C)}{\mathrm{vol}(T)}\right).\end{aligned}$$

It turns out that the described process of choosing each point y_i' allows us to give a bound on the last summand $\frac{l\mathrm{vol}(C)}{\mathrm{vol}(T)}$ in terms of α and the density of $\mathcal{C}$. We substitute this bound in the last inequality and optimize the resulting bound on $\theta_T(C)$ over all α in order to obtain the first part of Theorem 1.2.

The proof of the second part of Theorem 1.2 follows a similar scheme. Choose a lattice Λ and finite point set X such that the arrangement $C+\Lambda+X$ is a covering close to the optimal and project everything on the torus $T = \mathbb{R}^d/\Lambda$. If $0 < \alpha < 1$ and $(1-\alpha)C + X$ is not a packing on T, then we are able to find a point $y_1 \in T$ such that $\alpha C + y_1$ belongs to the intersection of at least two sets $C + x_1$, $C + x_1'$

for $x_1, x_1' \in T$. Then we define $X_1 = X \backslash \{x_1'\}$. As before, we iterate our process until we obtain that $(1-\alpha)C + X_l$ is a packing on T. In this case it is not always true that $(1-\alpha)C + \Lambda + X_l$ is a packing in $\mathbb{R}^d$, but this can be easily overcome under an additional assumption on the choice of Λ. Thus, we obtain

$$\begin{aligned}\delta_T(C) &= \delta_T((1-\alpha)\,C) \geq \operatorname{den}((1-\alpha)\,C + X_l + \Lambda) \\ &= \frac{|X_l|\,(1-\alpha)^d \operatorname{vol}(C)}{\operatorname{vol}(T)} = (1-\alpha)^d \left(\operatorname{den}(\mathcal{C}) - \frac{l\operatorname{vol}(C)}{\operatorname{vol}(T)}\right).\end{aligned}$$

Then the second part of Theorem 1.2 is obtained by bounding $\frac{l\operatorname{vol}(C)}{\operatorname{vol}(T)}$ and optimizing the last inequality in terms of α.

Acknowledgment. I would like to thank Ivan Izmestiev, Alexandr Polyanskii and the anonymous referees for their useful remarks.

References

1. Brass P., Moser W. and Pach J., *Research Problems in Discrete Geometry*, Springer, New York, 2005.
2. Dowker C. H., *On minimum circumscribed polygons*, Bull. Amer. Math. Soc. **50** (1944), 120–122.
3. Fejes Tóth, G., *A note on covering by convex bodies*, Canad. Math. Bull. **52** (2009), 361–365.
4. Fejes Tóth G. and Kuperberg. W., *A survey of recent results in the theory of packing and covering*, in: New trends in discrete and computational geometry, Algorithms Combin. vol. 10, Springer, 1993, 251–279.
5. Fejes Tóth G. and Kuperberg. W., *Packing and covering with convex sets*, in: Handbook of Convex Geometry, Elsevier, 1993, 799–860.
6. Fejes Tóth L., *Some packing and covering theorems*, Acta Sci. Math. (Szeged) **12** (1950), 62–67.
7. Ismailescu D., *Inequalities between lattice packing and covering densities of centrally symmetric plane convex bodies*, Discrete Comput. Geom. **25** (2001), 365–388.
8. Ismailescu D. and Kim B., *Packing and covering with centrally symmetric convex disks*, Discrete Comput. Geom. **51** (2014), 495–508.
9. Kuperberg W., *An inequality linking packing and covering densities of plane convex bodies*, Geom. Dedicata **23** (1987), 59–66.
10. Kuperberg W., *The set of packing and covering densities of convex disks*, Discrete Comput. Geom. **50** (2013), 1072–1084.
11. Prosanov R., *On a relation between packing and covering densities of convex bodies*, `arXiv:1811.03983`.
12. Rogers C. A., *Packing and Covering*, Cambridge Tracts in Mathematics and Mathematical Physics 54, Cambridge University Press, 1964.
13. Schmidt W. M., *Zur Lagening kongruenter Körper im Raum*, Monatsh. Math. **65** (1961), 154–158.
14. Schmidt W. M., *On the Minkowski-Hlawka theorem*, Illinois J. Math. **7** (1963), 18–23.
15. Smith E., *An improvement of an inequality linking packing and covering densities in* 3*-space*, Geom. Dedicata **117** (2006), 11–18.
16. Toth, G., *Measures of Symmetry for Convex Sets and Stability*, Universitext, Springer, 2015.

R. Prosanov, Université de Fribourg, Fribourg, Switzerland;

Moscow Institute of Physics and Technology, Dolgoprudny, Russia,

e-mail: `rprosanov@mail.ru`