Acta Math. Univ. Comenianae
Vol. LXXXVIII, 3 (2019), pp. 489–494

ASYMPTOTICALLY GOOD LOCAL LIST EDGE COLOURINGS

M. BONAMY, M. DELCOURT, R. LANG and L. POSTLE

Abstract. We study list edge colourings under local conditions. Our main result is an analogue of Kahn's theorem in this setting. More precisely, we show that, for a simple graph G with sufficiently large maximum degree Δ and minimum degree $\delta \geq \ln^{25} \Delta$, the following holds. Suppose that lists of colours $L(e)$ are assigned to the edges of G, such that, for each edge $e = uv$,

$$|L(e)| \geq (1 + o(1)) \cdot \max\{\deg(u), \deg(v)\}.$$

Then there is an L-edge-colouring of G. We also provide extensions of this result for hypergraphs and correspondence colourings, a generalization of list colouring.

1. Introduction

Consider a simple graph $G = (V, E)$ and an assignment of *lists of colours* $L(e) \subseteq \mathbb{N}$ to the edges of G. We say that a mapping of colours to the edges of G is an *L-colouring*, if every edge receives a colour from its list and no two adjacent edges receive the same colour. A classic problem in edge colouring, introduced independently by Erdős, Rubin, and Taylor [**5**]; and Vizing [**14**], is to determine lower bounds for $|L(e)|$ that guarantee that there is an L-colouring for all lists $L(e)$ satisfying these conditions.

A great deal of research in this area has focused on global bounds, i.e. where all lists are bounded from below by the same parameter. More precisely, the *list edge chromatic number*, denoted by $\mathrm{ch}'(G)$, is defined as the least k such that $\min|L(e)| \geq k$ guarantees an L-colouring. Some simple bounds are readily obtained. We denote the maximum degree of G by Δ. Then the condition $\min_{e\in E}|L(e)| \geq 2\Delta - 1$ yields the existence of an L-colouring, as the edges can be coloured greedily from the lists. Similarly, $\min_{e\in E}|L(e)| \geq \Delta$ is a trivial necessary condition for the existence of an L-colouring. We can extend this to $\min_{e\in E}|L(e)| \geq \chi'(G)$ by taking all lists to be the same, where $\chi'(G)$ denotes the chromatic index of G. Interestingly, we do not know of any graph for which the last bound does not hold with equality. The famous *List (Edge) Colouring Conjecture* states that this is not a coincidence.

Conjecture 1.1. Every graph G satisfies $\mathrm{ch}'(G) = \chi'(G)$.

Received June 5, 2019.
2010 *Mathematics Subject Classification.* Primary 05C15.
Key words and phrases. list coloring, edge coloring, correspondence coloring.

According to Jensen and Toft [**8**], the conjecture was suggested independently by Vizing, Albertson, Collins, Erdős, Tucker, and Gupta in the late seventies. It is known to be true for certain families of graphs, such as bipartite graphs by Galvin (Galvin's theorem) [**6**] and complete graphs of even and prime order by Häggvist and Jannsen [**7**] and Schauz [**12**], respectively. Moreover, Kahn showed that Conjecture 1.1 holds asymptotically, i.e. for all $\varepsilon > 0$ and every graph G with sufficiently large maximum degree Δ satisfies $\mathrm{ch}'(G) \leq (1+\varepsilon)\Delta$ [**9**].

Here, we study list edge colourings under local conditions, i.e. where $|L(e)|$ is lower bounded by a function that takes into account the local structure around e. For vertex colourings, such local notions were considered by Erdős, Rubin, and Taylor [**5**] and have been recently studied in terms of local clique sizes by Bonamy, Kelly, Nelson, and Postle [**1**] and for triangle-free graphs by Davies, de Joannis de Verclos, Kang, and Pirot [**3**]. Observe that we can refine the above bounds on $|L(e)|$ for $e = uv$, showing that $|L(e)| \leq \deg(u) + \deg(v) - 1$ is a sufficient condition for the existence of an L-colouring, while $|L(e)| \geq \deg(e) := \max\{\deg(u), \deg(v)\}$ is a necessary condition. Similar to the global setting, there is evidence that the bound $|L(e)| \geq \deg(e)$ is not far from being sufficient as well. Borodin, Kostochka, and Woodall showed that any bipartite graph admits an L-colouring provided that $|L(e)| \geq \deg(e)$, proving a local version of Galvin's theorem [**2**]. Moreover, they proved that for general graphs a bound of $|L(e)| \geq \deg(e) + \min\{\deg(u), \deg(v)\}$ suffices. Our main result is a local analogue of Kahn's theorem under the condition that the maximum degree is not too large in terms of the minimum degree.

Theorem 1.2. *For $\varepsilon > 0$, let G be a graph with sufficiently large maximum degree Δ, minimum degree $\delta \geq \ln^{25} \Delta$, and lists of colours $L(e)$. If $|L(e)| \geq (1+\varepsilon)\deg(e)$ for every edge $e \in E$, then there is an L-colouring of G.*

We note that results similar to Theorem 1.2 have been obtained for correspondence colourings by Molloy [**10**] and hypergraphs by Molloy and Reed [**11**] in the global setting. We prove Theorem 1.2 in a more general setting involving weighted lists of colours and also covering linear hypergraphs and correspondence colourings. To state this, we need a few further definitions.

Let $G = (V, E)$ be a k-uniform hypergraph. We denote incident edges by $e \sim f$. We say that G is *linear*, if any two edges intersect in at most one vertex. An *edge correspondence* σ of G consists of integer permutations $\sigma_{e,f} = \sigma_{f,e}$ for all edges $e \sim f$. For edges $e \sim f$ and colours $c, c' \in \mathbb{N}$, we say that (e, c) *blocks* (f, c'), if $\sigma_{e,f}(c) = c'$. An assignment of *weights* to the edges is a function $\mu\colon E \times \mathbb{N} \to [0,1]$. We write $|A|_\mu = \sum_{(e,c)\in A} \mu(e,c)$ for a set $A \subseteq E \times \mathbb{N}$. An (μ,σ)-colouring is a function $\gamma\colon E \to \mathbb{N}$ such that

- $\mu(e, \gamma(e)) > 0$ for every $e \in E$; and
- $(e, \gamma(e))$ does not block $(f, \gamma(f))$ for all edges $e \sim f$.

We denote by $\mathcal{L}(e)$ the set of all pairs (e, c) for $c \in \mathbb{N}$. Finally, let $\mathcal{N}_{G,\sigma}(e, v, c)$ be the set containing all pairs $(f, c') \in E \times \mathbb{N}$ such that f is incident to v, $f \neq e$ and (f, c') blocks (e, c). If σ is the trivial correspondence, i.e. where all permutations are identities, we write $\mathcal{N}_G(e, v, c) = \mathcal{N}_{G,\sigma}(e, v, c)$ and μ-colouring in place of (μ, σ)-colouring.

Now we are ready to state our (stronger) main result.

Theorem 1.3. *For every $k \in \mathbb{N}$ and $\varepsilon > 0$ there is δ with the following properties. Let $G = (V, E)$ be a k-uniform linear hypergraph with an edge correspondence σ and weights μ. Suppose that*

(a) $\mu(e, c) \in \{0\} \cup \left[\exp\left(-\delta^{1/25}\right), \delta^{-1}\right]$*; and*
(b) $|\mathcal{L}(e)|_\mu \geq (1+\varepsilon) \cdot |\mathcal{N}_{G,\sigma}(e, v, c)|_\mu$

for every edge $e \in E$, vertex $v \in e$ and colour $c \in \mathbb{N}$. Then there is a (μ, σ)-colouring of G.

In the remainder of the abstract, we will outline the proofs of our results.

2. Proof of the main result

We start by showing how Theorem 1.2 can be derived from Theorem 1.3.

Proof of Theorem 1.2. Let G, δ, Δ and $L(e)$ be as in the statement of Theorem 1.2. We assign weights $\mu(e, c) = \frac{1}{\deg(e)}$ to each edge $e \in E$ and colour $c \in L(e)$. Whenever $c \notin L(e)$, we set $\mu(e, c) = 0$. Note that, for each edge $e \in E$ and vertex $v \in e$, we have

$$|\mathcal{L}(e)|_\mu \geq \tfrac{|L(e)|}{\deg(e)} \geq 1 + \varepsilon$$

and

$$|\mathcal{N}_G(e, v, c)|_\mu = \sum_{f \sim v, f \neq e} \frac{1}{\deg(f)} \leq \sum_{f \sim v, f \neq e} \frac{1}{\deg(v)} \leq 1.$$

So in particular, we can bound $|\mathcal{L}(e)|_\mu \geq (1+\varepsilon) \cdot |\mathcal{N}_G(e, v, c)|_\mu$. Thus we can apply Theorem 1.3 with $k = 2$ and σ being the trivial correspondence to obtain an L-colouring. □

Next, we sketch the proof of Theorem 1.3. Suppose that G, μ, ε are as in the statement and, for sakes of simplicity, $k = 2$ and σ is the trivial correspondence. An application of the (general) Lovász Local Lemma [**4**] shows, that we can find a μ-colouring, provided that $|\mathcal{L}(e)|_\mu \geq 9 \cdot |\mathcal{N}_G(e, v, c)|_\mu$ for every edge $e \in E$, vertex $v \in e$, and colour $c \in \mathbb{N}$.

To obtain the desired factor of $1 + \varepsilon$ instead 9, we follow an iterative approach. In each step, we colour a few further edges improving the above factor for the remainder of the graph by a factor of roughly $1 + \frac{\varepsilon}{\ln \delta}$. Hence, after $O(\log \delta)$ iterations, we can finish by applying the local lemma. The next lemma formalizes this discussion and presents the heart of our proof.

Lemma 2.1. *Let $\varepsilon > 0$ and $\delta \in \mathbb{N}$ large enough. Let $G = (V, E)$ be a graph with weights μ. Suppose that there are $\ell, n \in N$ such that*

(a) $\mu(e, c) \in \{0\} \cup \left[\exp\left(-\delta^{1/20}\right), \delta^{-1}\right]$*;*
(b) $\frac{\ell}{n} \geq 1 + \varepsilon$*;*
(c) $|\mathcal{L}(e)|_\mu \geq \ell$*; and*
(d) $|\mathcal{N}_G(e, v, c)|_\mu \leq n$*;*

for every edge $e \in E$, vertex $v \in e$ and colour $c \in \mathbb{N}$.

It follows that there is a partial μ-colouring of G with the following properties. Let E' be the set of uncoloured edges and $G' = (V, E')$. For each $e \in E'$, let $\mu'(e, \cdot)$ be obtained from $\mu(e, \cdot)$ by setting $\mu'(e, c) = 0$, for each $c \in \mathbb{N}$, if an edge adjacent to e is coloured c. Then, there are $\ell', n' \in \mathbb{N}$ such that

(a) $\frac{\ell'}{n'} \geq \left(1 + \frac{\varepsilon}{2 \ln \delta}\right) \frac{\ell}{n}$;
(b) $|\mathcal{L}(e)|_{\mu'} \geq \ell'$; *and*
(c) $|\mathcal{N}_{G'}(e, v, c)|_{\mu'} \leq n'$;

for every edge $e \in E'$, vertex $v \in e$ and colour $c \in \mathbb{N}$.

We remark that Lemma 2.1 is a simplified version of the actual lemma, omitting hypergraphs, correspondences, and ignoring some of the additional properties that are necessary for tracking the sizes of the involved parameters during the iterations.

3. The naive colouring procedure

In the following we will provide some more details on the proof of Lemma 2.1. We find the desired colouring using the *naive colouring procedure* introduced by Kahn [**9**]. This method consists (in its simplest form) of two steps. In step (I), we randomly assign to each edge a small (possibly empty) set of permissible colours from its list independently from all other edges. The colours on these lists are candidates for the final colouring. However, some of these candidates might be in conflict with each other, i.e. the lists of two adjacent edges may contain the same colour. In step (II), we resolve these conflicts by removing some colours from the assigned sets. We then obtain a partial μ-colouring by assigning to each edge with a non-empty list an arbitrary colour from its list. A concentration analysis shows that with positive probability this μ-colouring has the desired properties.

Let $\varepsilon > 0$, δ, G and μ be as in the statement of Lemma 2.1. Without loss of generality, we can assume that $|\mathcal{L}(e)|_\mu = \ell$ for every edge. (Otherwise, we simply 'remove' some colours of positive weight.) We denote $\mathrm{Keep} = 1 - \frac{n}{\ell} \frac{1}{\ln \delta}$.

We use the following random colouring procedure to colour some of the edges of G. Initialize $\mu'(e, \cdot)$ as copy of $\mu(e, \cdot)$ for each $e \in E$.

(I) For every colour $c \in \mathbb{N}$ and edge $e \in E$, assign c to e with probability $\frac{\mu(e,c)}{\ell} \frac{1}{\ln \delta}$ independently of all other assignments.
(II) For each edge e, vertex $v \in e$ and every pair $(f, c) \in \mathcal{N}(e, v, c)$, if c was assigned to f, then
 (a) set $\mu'(e, c) = 0$; and
 (b) if c was assigned to e, remove c from e.

Let E' be the set of uncoloured edges after step (II) and $G' = (V, E')$. In the following, we analyse the μ'-weighted sizes of $\mathcal{L}(e)$ and $\mathcal{N}_{G'}(e, v, c)$. We claim that with positive probability

(a) $|\mathcal{L}(e)|_{\mu'} \geq \ell' := \ell \cdot \mathrm{Keep}^2 - \delta^{2/3}$; and
(b) $|\mathcal{N}_{G'}(e, v, c)|_{\mu'} \leq n' := n \cdot \mathrm{Keep} \cdot \left(1 - \frac{\mathrm{Keep}^2}{\ln \delta}\right) + \delta^{2/3}$

for every edge $e \in E'$, vertex $v \in e$, and colour $c \in \mathbb{N}$. Computations show that (a) and (b) together imply $|\mathcal{L}(e)|_{\mu'} \geq \left(1 + \frac{\varepsilon}{2\ln\delta}\right)(1+\varepsilon)|\mathcal{N}_{G'}(e,v,c)|_{\mu'}$, i.e. the desired outcome of Lemma 2.1. So it only remains to prove these claims.

The strategy is to first argue that (a) and (b) are satisfied with high probability for a fixed edge $e \in E'$, vertex $v \in e$, and colour $c \in \mathbb{N}$. We then use the local lemma to show that (a) and (b) hold uniformly for all e, v, and c. Let us remark that the bound $\mu(e) \geq \exp\left(-\delta^{1/20}\right)$ is crucial for the last step, as it allows us to bound the size of the second, third, and fourth neighbourhoods.

When bounding $|\mathcal{L}(e)|_{\mu'} \geq \ell'$ and $|\mathcal{N}_{G'}(e,v,c)|_{\mu'} \leq n'$ for fixed $e \in E'$, $v \in e$, and $c \in \mathbb{N}$, our approach follows a concentration argument. If $\mu(e,c) > 0$ for colour $c \in \mathbb{N}$, then the probability that $\mu'(e,c) = \mu(e,c)$ is roughly[1] Keep^2. It follows that $\mathbf{E}\left(|\mathcal{L}(e)|_{\mu'}\right) \approx |\mathcal{L}(e)|_{\mu} \cdot \mathrm{Keep}^2 = \ell \cdot \mathrm{Keep}^2$. Using a weighted version of Chernoff's bound, we then show that $|\mathcal{L}(e)|_{\mu'}$ is highly concentrated around its expectation. The case of $|\mathcal{N}_{G'}(e,v,c)|_{\mu'}$, is treated in a similar way. However, the analysis is more complicated, in particular with regards to correspondences, and requires the use of Talagrand's inequality [**13**].

4. Conclusion

We conclude this abstract with a few open problems. An obvious improvement to our results would be to remove the condition $\delta \geq \ln^{25}\Delta$ in Theorem 1.2. It does play a critical role in our proof and it seems therefore that new ideas are required to solve this.

One way to interpret Conjecture 1.1, is to say that list edge colouring is hardest, when all lists are the same. Given our results, one might wonder, if we have a similar phenomenon in the local setting. We define $\chi'_{\mathrm{loc}}(G)$ to be the smallest k such that there is an L-colouring from the lists $L(e) = \{1,\ldots,\deg(e)+k\}$. Similarly, we let $\mathrm{ch}'_{\mathrm{loc}}(G)$ be the smallest k such that there is an L-colouring for every assignment of lists $|L(e)| \geq \deg(e)+k$. Note that for a regular graph G with $\chi'(G) \geq \Delta+1$, such k would clearly be at least 1. In the light of Conjecture 1.1, it is natural to ask if there are any graphs with $\mathrm{ch}'_{\mathrm{loc}}(G) \neq \chi'_{\mathrm{loc}}(G)$. This is probably not easy to answer. On the other hand, we do not know of any graph for which $\chi'_{\mathrm{loc}}(G) > 1$. It would be interesting (and likely more feasible) to find such a graph or even a sequence of graphs for which $\chi'_{\mathrm{loc}}(G)$ is unbounded.

References

1. Bonamy M., Kelly T., Nelson P. and Postle L., *Bounding χ by a fraction of Δ for graphs without large cliques*, `arXiv:1803.01051`.
2. Borodin O. V., Kostochka A. V. and Woodall D. R., *List edge and list total colourings of multigraphs*, J. Combin. Theory Ser. B **71** (1997), 184–204.
3. Davies E., de Joannis de Verclos R., Kang R. J., and Pirot F., *Colouring triangle-free graphs with local list sizes*, `arXiv:1812.01534`.

[1]By adding a simple balancing step at the end of the procedure we can get equality here.

4. Erdős P. and Lovász L., *Problems and results on finite and infinite graphs*, in: Recent Advances in Graph Theory (Proc. Second Czechoslovak Sympos., Prague, 1974), 1975, 183–192.
5. Erdős P., Rubin A. L. and Taylor H., *Choosability in graphs*, in: Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium, vol. 26, 1979, 125–157.
6. Galvin F., *The list chromatic index of a bipartite multigraph*, J. Combin. Theory Ser. B **63** (1995), 153–158.
7. Häggkvist R. and Janssen J., *New bounds on the list-chromatic index of the complete graph and other simple graphs*, Combin. Probab. Comput. **6** (1997), 295–313.
8. Jensen T. R. and Toft B., *Graph Coloring Problems*, vol. 39. John Wiley & Sons, 1995.
9. Kahn J., *Asymptotically good list-colorings*, J. Combin. Theory Ser. A **73** (1996), 1–59.
10. Molloy M., *Asymptotically good edge correspondence colouring*, `arXiv:1808.08594`.
11. Molloy M. and Reed B., *Near-optimal list colorings*, Random Structures Algorithms **17** (2000), 376–402.
12. Schauz U. *Proof of the list edge coloring conjecture for complete graphs of prime degree*, Electron. J. Combin. **21** (2014), 3–43.
13. Talagrand M., *Concentration of measure and isoperimetric inequalities in product spaces*, Publications Mathématiques de l'IHÉS **81** (1995), 73–205.
14. Vizing V. G., *The chromatic class of a multigraph*, Kibernetika **1** (1965), 29–39, English translation in: Cybernet. Systems Anal. **1** (1965), 32–41.

M. Bonamy, Université de Bordeaux, France,
e-mail: `marthe.bonamy@u-bordeaux.fr`

M. Delcourt, University of Waterloo, Waterloo, Canada,
e-mail: `michelle.delcourt@uwaterloo.ca`

R. Lang, University of Waterloo, Waterloo, Canada,
e-mail: `r7lang@uwaterloo.ca`

L. Postle, University of Waterloo, Waterloo, Canada,
e-mail: `lpostle@uwaterloo.ca`