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MINOR-OBSTRUCTIONS FOR APEX SUB-UNICYCLIC GRAPHS

A. LEIVADITIS, A. SINGH, G. STAMOULIS, D. M. THILIKOS, K. TSATSANIS
AND V. VELONA

Abstract. A graph is *sub-unicyclic* if it contains at most one cycle. We also say that a graph G is *k-apex sub-unicyclic* if it can become sub-unicyclic by removing k of its vertices. We identify 29 graphs that are the minor-obstructions of the class of 1-apex sub-unicyclic graphs, i.e., the set of all minor minimal graphs that do not belong in this class. For bigger values of k, we give an exact structural characterization of all the cactus graphs that are minor-obstructions of k-apex sub-unicyclic graphs and we enumerate them. This implies that, for every k, the class of k-apex sub-unicyclic graphs has at least $0.34 \cdot k^{-2.5}(6.278)^k$ minor-obstructions.

1. Introduction

A graph is called *unicyclic* if it contains exactly one cycle and is called *sub-unicyclic* if it contains at most one cycle. Notice that sub-unicyclic graphs are exactly the subgraphs of unicyclic graphs.

A graph H is a minor of a graph G if a graph isomorphic to H can be obtained by some subgraph of G after a series of contractions. We say that a graph class $\mathcal{G}$ is *minor-closed* if every minor of every graph in $\mathcal{G}$ also belongs in $\mathcal{G}$. We also define $\mathbf{obs}(\mathcal{G})$, called the *minor-obstruction set* of $\mathcal{G}$, as the set of minor-minimal graphs not in $\mathcal{G}$. It is easy to verify that if $\mathcal{G}$ is minor-closed, then $G \in \mathcal{G}$ iff G excludes all graphs in $\mathbf{obs}(\mathcal{G})$ as a minor. Because of Robertson and Seymour Theorem [**17**], $\mathbf{obs}(\mathcal{G})$ is finite for every minor-closed graph class. That way, $\mathbf{obs}(\mathcal{G})$ can be seen as a *complete characterization* of $\mathcal{G}$ via a finite set of forbidden graphs. The identification of $\mathbf{obs}(\mathcal{G})$ for distinct minor-closed classes has attracted a lot of attention in Graph Theory.

There are several ways to construct minor-closed graph classes from others. A popular one is to consider the set of all *k-apices* of a graph class $\mathcal{G}$, denoted by

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$\mathcal{A}_k(\mathcal{G})$, that contains all graphs that can give a graph in $\mathcal{G}$, after the removal of at most k vertices. It is easy to verify that if $\mathcal{G}$ is minor closed, then the same holds for $\mathcal{A}_k(\mathcal{G})$ as well, for every non-negative integer k. It was also proved in [**1**] that the construction of $\mathbf{obs}(\mathcal{A}_k(\mathcal{G}))$, given $\mathbf{obs}(\mathcal{G})$ and k, is a computable problem.

A lot of research has been oriented to the (partial) identification of the minor-obstructions of the k-apices, of several minor-closed graph classes. For instance, $\mathbf{obs}(\mathcal{A}_k(\mathcal{G}))$ has been identified for $k \in \{1, \ldots, 7\}$ when $\mathcal{G}$ is the set of edgeless graphs [**3, 8, 7**], and for $k \in \{1, 2\}$ when $\mathcal{G}$ is the set of acyclic graphs [**6**]. Recently, $\mathbf{obs}(\mathcal{A}_1(\mathcal{G}))$ was identified when $\mathcal{G}$ is the class of outerplanar graphs [**4**] and when $\mathcal{G}$ is the class of cactus graphs (as announced in [**11**]). A particularly popular problem is identification of $\mathbf{obs}(\mathcal{A}_k(\mathcal{G}))$ when $\mathcal{G}$ is the class of planar graphs. The best advance on this question was done recently by Jobson and Kzdy [**14**] who identified all 2-connected minor-obstructions of 1-apex planar graphs. Another recent result is the identification of $\mathbf{obs}(\mathcal{A}_1(\mathcal{P}))$ where $\mathcal{P}$ is the class of all pseudoforests, i.e., graphs where all connected components are sub-unicyclic [**16**].

A different direction is to upper-bound the size of the graphs $\mathbf{obs}(\mathcal{A}_k(\mathcal{G}))$ by some function of k. In this direction, it was proved in [**13**] that the size of the graphs in $\mathbf{obs}(\mathcal{A}_k(\mathcal{G}))$ is bounded by a polynomial on k in the case where the $\mathbf{obs}(\mathcal{G})$ contains some planar graph (see also [**19**]). Another line of research is to prove lower bounds to the size of $\mathbf{obs}(\mathcal{A}_k(\mathcal{G}))$. In this direction Michael Dinneen proved in [**5**] that, if all graphs in $\mathbf{obs}(\mathcal{G})$ are connected, then $|\mathbf{obs}(\mathcal{A}_k(\mathcal{G}))|$ is exponentially big.

Another way to prove lower bounds to $|\mathbf{obs}(\mathcal{A}_k(\mathcal{G}))|$ is to completely characterize, for every k, the set $\mathbf{obs}(\mathcal{A}_k(\mathcal{G})) \cap \mathcal{H}$, for some graph class $\mathcal{H}$, and then lower bound $|\mathbf{obs}(\mathcal{A}_k(\mathcal{G}))|$ by counting (asymptotically or exactly) all the graphs in $\mathbf{obs}(\mathcal{A}_k(\mathcal{G})) \cap \mathcal{H}$. This last approach has been applied in [**18**] when $\mathcal{G}$ is the class of acyclic graphs and $\mathcal{H}$ is the class of outerplanar graphs (see also [**10, 15**]).
Our results. In this paper we study the set $\mathbf{obs}(\mathcal{A}_k(\mathcal{S}))$ where $\mathcal{S}$ is the class of sub-unicyclic graphs. Certainly the class $\mathcal{S}$ is minor-closed (while this is not the case for unicyclic graphs). It is easy to see that $\mathbf{obs}(\mathcal{S}) = \{2K_3, K_4^-, Z\}$, where $2K_3$ is the disjoint union of two triangles, K_4^- is the complete graph on 4 vertices minus an edge, and Z the *butterfly graph*, obtained by $2K_3$ after identifying two vertices of its triangles (we call the result of this identification *central vertex* of Z).

Our first result is the identification of $\mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$, i.e., the minor-obstruction set of all 1-apices of sub-unicyclic graphs (Section 2). This set contains 29 graphs that is the union of two sets $\mathcal{L}_0$ and $\mathcal{L}_1$, depicted in Figures 1 and 2 respectively. An important ingredient of our proof is the notion of a *nearly-biconnected graph*, that is any graph that is either biconnected or it contains only one cut-vertex joining two blocks where one of them is a triangle. We first prove that $\mathcal{L}_0$ is the set of minor-obstructions in $\mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$ that are not nearly-biconnected. The proof is completed by proving that the nearly-biconnected graphs in $\mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$ are also minor-obstructions for 1-apex pseudoforests, i.e., members of $\mathbf{obs}(\mathcal{A}_1(\mathcal{P}))$. As this set is known from [**16**], we can identify the remaining obstructions in $\mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$, that is the set $\mathcal{L}_1$, by exhaustive search.

Our second result is an exponential lower bound on the size $\mathbf{obs}(\mathcal{A}_k(\mathcal{S}))$ (Section 3). For this we completely characterize, for every k, the set $\mathbf{obs}(\mathcal{A}_k(\mathcal{S})) \cap \mathcal{K}$ where $\mathcal{K}$ is the set of all cacti (graphs whose all blocks are either edges or cycles). In particular, we first prove that each connected cactus obstruction in $\mathbf{obs}(\mathcal{A}_k(\mathcal{S}))$ can be obtained by identifying non-central vertices of $k+1$ butterfly graphs and then we give a characterization of disconnected cacti in $\mathbf{obs}(\mathcal{A}_k(\mathcal{S}))$ in terms of obstructions in $\mathbf{obs}(\mathcal{A}_{k'}(\mathcal{S}))$ for $k' < k$ (we stress that here the result of Dinneen in [**5**] does not apply immediately, as not all graphs in $\mathbf{obs}(\mathcal{S})$ are connected).

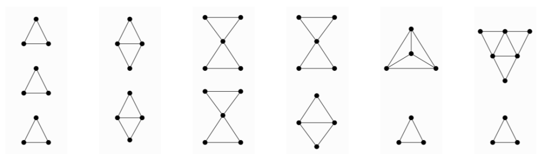
After identifying $\mathbf{obs}(\mathcal{A}_k(\mathcal{S})) \cap \mathcal{K}$, the next step is to count the number of its elements (Section 4). To that end, we employ the framework of the *Symbolic Method* and *Singularity Analysis* that was developed in [**12**]. The combinatorial construction that we devise relies critically on the *Dissymmetry Theorem for Trees*, by which one can move from the enumeration of rooted tree structures to unrooted ones. Our estimation is

$$|\mathbf{obs}(\mathcal{A}_k(\mathcal{S})) \cap \mathcal{K}| \sim c \cdot k^{-5/2} \cdot x^k,$$

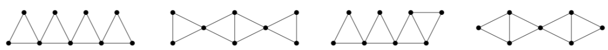
where $c \approx 0.33995$ and $x \approx 6.27888$. This provides an exponential lower bound for $|\mathbf{obs}(\mathcal{A}_k(\mathcal{S}))|$.

2. Minor-obstructions for apex sub-unicyclic graphs

In this section we identify the set $\mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$. Part of it will be the set $\mathcal{L}_0$ containing the graphs depicted in Figure 1.



(a) O_1^0 (b) O_2^0 (c) O_3^0 (d) O_4^0 (e) O_5^0 (f) O_6^0



(g) O_1^1 (h) O_2^1 (i) O_3^1 (j) O_4^1

Figure 1. The set $\mathcal{L}_0$ of obstructions for $\mathcal{A}_1(\mathcal{S})$ that are not nearly-biconnected.

Lemma 2.1. *If $G \in \mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$ and G is not connected, then it holds that $G \in \{O_1^0, \ldots, O_6^0\}$.*

Lemma 2.2. *If G is a connected graph in $\mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$, with at least three cut-vertices, then $G \cong O_1^1$.*

Lemma 2.3. *If G is a connected graph in $\mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$ with exactly two cut-vertices, then $G \in \{O_2^1, O_3^1\}$.*

Nearly-biconnected graphs. We say that a graph G is *nearly-biconnected* if it is either biconnected or it contains exactly one cut-vertex x and contains two blocks where one of them is a triangle.

Lemma 2.4. *Let $G \in \mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$ be a connected graph that contains exactly one cut-vertex. Then either $G \cong O^1_4$ or G is nearly-biconnected.*

We need the following fact:

Fact 2.5. *The graphs in $\mathbf{obs}(\mathcal{A}_1(\mathcal{P}))$ that are nearly-biconnected and belong in $\mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$ are the graphs in Figure 2.*

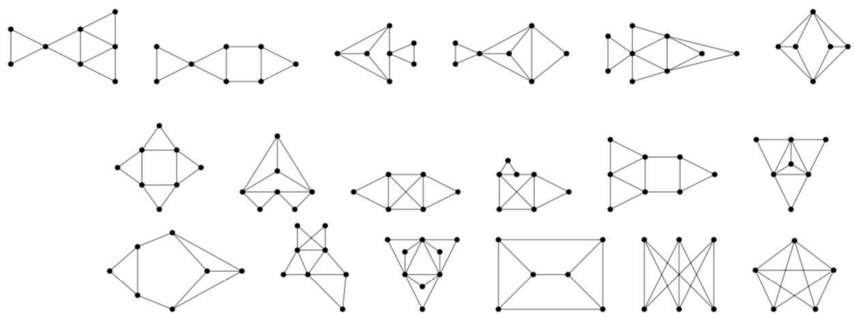


Figure 2. The set $\mathcal{L}_1$ of the 19 nearly-biconnected minor-obstructions for $\mathcal{A}_1(\mathcal{S})$ that are also obstructions for $\mathcal{A}_1(\mathcal{P})$.

The set $\mathbf{obs}(\mathcal{A}_1(\mathcal{P}))$ consists of 33 graphs and has been identified in [**16**]. The correctness of Fact 2.5 can be verified by exhaustive check, considering all nearly-biconnected graphs in $\mathbf{obs}(\mathcal{A}_1(\mathcal{P}))$ (they are 26) and then filter those that belong in $\mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$. For this, one should pick those that become apex sub-unicyclic after the contraction or removal of each of their edges. Notice that the fact that these graphs are not apex-sub-unicyclic follows directly by the fact that they are not apex-pseudoforests (as members of $\mathbf{obs}(\mathcal{A}_1(\mathcal{P}))$) and the fact that $\mathcal{S} \subseteq \mathcal{P}$. The choice of $\mathcal{L}_1$ is justified by the next lemma.

Lemma 2.6. *If G is a nearly-biconnected graph in $\mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$, then $G \in \mathbf{obs}(\mathcal{A}_1(\mathcal{P}))$.*

We are now ready to prove the main result of this section.

Theorem 2.7. $\mathbf{obs}(\mathcal{A}_1(\mathcal{S})) = \mathcal{L}_0 \cup \mathcal{L}_1$.

Proof. Recall that $\mathcal{L}_0 = \{O^0_1, \dots, O^0_6\} \cup \{O^1_1, O^1_2, O^1_3, O^1_4\}$. Notice that $\mathcal{L}_0 \cup \mathcal{L}_1 \subseteq \mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$. Let $G \in \mathbf{obs}(\mathcal{A}_1(\mathcal{S}))$. If G is disconnected, then, from Lemma 2.1, $G \in \{O^0_1, \dots, O^0_6\}$. If G is connected and has at least three cut-vertices, then from Lemma 2.2, $G \cong O^1_1$. If G is connected and has exactly two cut-vertices, then from Lemma 2.3, $G \in \{O^1_2, O^1_3\}$. If G is connected with exactly one a cut-vertex and is not nearly-biconnected then, from Lemma 2.4, $G \cong O^1_4$. We just proved that if G is not nearly-biconnected, then $G \in \mathcal{L}_0$. On the other side, if G is nearly-biconnected, then from Lemma 2.6, $G \in \mathbf{obs}(\mathcal{A}_1(\mathcal{P}))$, therefore, from Fact 2.5, $G \in \mathcal{L}_1$, as required. □

3. Structural characterisation of cactus obstructions

Recall that a *cactus graph* is a graph where all its blocks are either edges or cycles. Equivalently, a graph is a cactus graph, if it does not contain K_4^- as a minor. We denote by $\mathcal{K}$ the set of all cactus graphs. In this section we provide a complete characterization of the class of $\mathbf{obs}(\mathcal{A}_k(\mathcal{S})) \cap \mathcal{K}$.
Butterflies and Butterfly-Cacti. We denote by Z the butterfly graph. We will frequently refer to graphs isomorphic to Z simply as *butterflies*. Given a butterfly Z we call all its four vertices that have degree two, *extremal vertices* of Z and the unique vertex of degree four, *central vertex*.

Let k be a positive integer. We recursively define the graph class of the *k-butterfly-cacti*, denoted by $\mathcal{Z}_k$, as follows: We set $\mathcal{Z}_1 = \{Z\}$, where Z is the butterfly graph, and given a $k \geq 2$ we say that $G \in \mathcal{Z}_k$ if there is a graph $G' \in \mathcal{Z}_{k-1}$ such that G is obtained if we take a copy of the butterfly graph Z and then we identify one of its extremal vertices with a non-central vertex of G'. The *central vertices* of the obtained graph G are the central vertices of G' and the central vertex of Z.

Theorem 3.1. *Let $k \in \mathbb{N}$, let G be a disconnected cactus graph in $\mathbf{obs}(\mathcal{A}_k(\mathcal{S}))$, and let $G_1, G_2, \ldots, G_r$ be the connected components of G. Then, one of the following holds:*

- $G \cong (k+2)K_3$
- *there is a sequence $k_1, k_2, \ldots, k_r$ such that for every $i \in [r]$, G_i is a graph in $\mathcal{Z}_{k_i}$ and $\sum_{i\in[r]} k_i = k+1$.*

4. Enumeration of cactus obstructions

Let $\mathcal{G} = \mathbf{obs}(\mathcal{A}_k(\mathcal{S})) \cap \mathcal{K}$. In this section, we determine the asymptotic growth of $g_k = |\mathbf{obs}(\mathcal{A}_k(\mathcal{S})) \cap \mathcal{K}|$ and $z_k = |\mathcal{Z}_k|$. To this end, we make use of the Symbolic Method framework and the corresponding analytic techniques, as developed in [**12**].

4.1. A bijection of $\mathcal{Z}$ with a family of trees

We begin by giving a bijection between the combinatorial class $\mathcal{Z}$, i.e., connected graphs in $\mathbf{obs}(\mathcal{A}_k(\mathcal{S})) \cap \mathcal{K}$ counted with respect to the number of butterfly-subgraphs, and the following family of trees. Let $\mathcal{T}$ be the family of trees having three different types of vertices, namely □-, △-, and ○-vertices, and meeting the following conditions:

1. The neighbourhood of a □-vertex consists of two △-vertices.
2. The neighbourhood of a △-vertex consists of a □-vertex and two ○-vertices.
3. The neighbourhood of a ○-vertex consists of one or more △-vertices.

Consider the combinatorial class $\mathcal{T}$ counted with respect to the number of its $\square$-vertices. Then, one can construct a bijection between $\mathcal{Z}$ and $\mathcal{T}$ that proves the following Lemma (see Figure 3 for an example).

Lemma 4.1. *The combinatorial classes $\mathcal{Z}$ and $\mathcal{T}$ are isomorphic.*

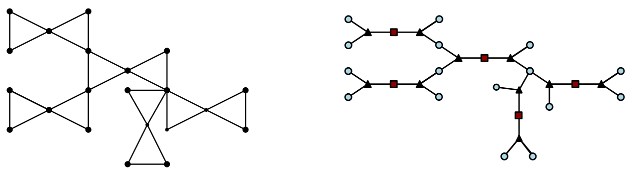


Figure 3. A graph in $\mathcal{Z}$ and its image in $\mathcal{T}$, under the bijection of Theorem 4.1.

By Lemma 4.1, enumerating $\mathcal{Z}$ is equivalent to enumerating $\mathcal{T}$. To this end, we will make use of the combinatorial classes $\mathcal{T}^{\square}, \mathcal{T}^{\triangle}, \mathcal{T}^{\circ}, \mathcal{T}^{\circ-\triangle}, \mathcal{T}^{\square-\triangle}, \mathcal{T}^{\square\to\triangle}, \mathcal{T}^{\triangle\to\square}, \mathcal{T}^{\triangle\to\circ}, \mathcal{T}^{\circ\to\triangle}$, which are trees in $\mathcal{T}$ that are rooted on vertices, edges, or directed edges, of the indicated types. We denote by plain letters the corresponding generating functions. Using the well-known Dissymmetry Theorem for Trees (see [**2**]), we can show the following:

Lemma 4.2. $G(x) = \exp\left(\sum_{k\geq 1} \frac{T(x^k)}{k}\right)$. *Moreover, $T(x)$ satisfies:*

$$T(x) = T^{\square}(x) + T^{\triangle}(x) + T^{\circ}(x) - T^{\square\to\triangle}(x) - T^{\triangle\to\circ}(x). \tag{1}$$

To obtain defining systems for $T^{\square}(x), T^{\triangle}(x), T^{\circ}(x), T^{\square\to\triangle}(x), T^{\triangle\to\circ}(x)$, we define the auxiliary combinatorial classes $\mathcal{T}_{\diamond}$ and $\mathcal{T}_{\star}$. $\mathcal{T}_{\diamond}$ contains trees in $\mathcal{T}$ rooted at a leaf and $\mathcal{T}_{\star}$ contains multisets of trees in $\mathcal{T}_{\diamond}$.

Lemma 4.3. *The generating functions $T_{\star}, T_{\diamond}, T^{\square}, T^{\triangle}, T^{\circ}, T^{\square\to\triangle}, T^{\triangle\to\circ}$ are defined through the following system of functional equations.*

$$\begin{aligned}
T_{\diamond}(x) &= \frac{x}{2}\exp\Big(\sum_{k\geq 1}\frac{T_{\diamond}(x^k)}{k}\Big)\Big(\exp\Big(\sum_{k\geq 1}\frac{2\,T_{\diamond}(x^k)}{k}\Big) + \exp\Big(\sum_{k\geq 1}\frac{T_{\diamond}(x^{2k})}{k}\Big)\Big)\\
T_{\star}(x) &= \exp\Big(\sum_{k\geq 1}\frac{T_{\diamond}(x^k)}{k}\Big),\quad T^{\circ}(x) = \exp\Big(\sum_{k\geq 1}\frac{T_{\diamond}(x^k)}{k}\Big) - 1\\
T^{\square}(x) &= \frac{x}{8}\,T_{\star}(x)^4 + \frac{x}{4}T_{\star}(x)^2T_{\star}(x^2) + \frac{3x}{8}\,T_{\star}(x^2)^2 + \frac{x}{4}\,T_{\star}(x^4)\\
T^{\triangle}(x) &= \frac{x}{4}\,T_{\star}(x)^4 + \frac{x}{2}\,T_{\star}(x)^2T_{\star}(x^2) + \frac{x}{4}T_{\star}(x^2)^2\\
T^{\square\to\triangle}(x) &= \frac{x}{4}\,T_{\star}(x)^4 + \frac{x}{2}\,T_{\star}(x)^2T_{\star}(x^2) + \frac{x}{4}T_{\star}(x^2)^2\\
T^{\triangle\to\circ}(x) &= \frac{x}{2}\,T_{\star}(x)^4 + \frac{x}{2}\,T_{\star}(x)^2\,T_{\star}(x^2)
\end{aligned}$$

By the defining systems of $T(x)$ and $G(x)$, we can obtain the first terms

$$T(x) = x + x^2 + 3\,x^3 + 7\,x^4 + 25\,x^5 + 88\,x^6 + 366\,x^7 + 1583\,x^8 + 7336\,x^9 + \cdots$$
$$G(x) = 1 + z + 2\,x^2 + 5\,x^3 + 13\,x^4 + 41\,x^5 + 143\,x^6 + 558\,x^7 + 2346\,x^8 + \cdots$$

4.2. Asymptotic analysis

Having set up a system of functional equations for the generating functions $Z(x)$ and $G(x)$, we can determine the asymptotic growth of z_k and g_k via the process of singularity analysis (see `http://www.cs.upc.edu/~sedthilk/osmc/apexmo.mw` for the corresponding numerical computations).

Lemma 4.4. *The generating functions $Z(x), G(x)$ have a unique singularity of smallest modulus at the same positive number $\rho < 1$. Moreover, they are analytic in a dented domain at ρ and satisfy expansions*

$$Z(x) = Z_0 + \sum_{k\geq 2} Z_k X^k, \quad G(x) = G_0 + \sum_{k\geq 2} G_k X^k, \quad \textit{where} \quad X = \sqrt{1 - x/\rho},$$

locally around ρ. The coefficients Z_i, G_i, and ρ are computable; in particular, $\rho \approx 0.15926$.

Proof Sketch. To analyse $T_\diamond(z)$, we use [**9**, Proposition 1, Lemma 1]. Define $F(x,y)$ to be the following expression: $\frac{x}{2}\exp(y + \sum_{k\geq 2} \frac{T_\diamond(x^k)}{k})(\exp(2y + \sum_{k\geq 2} \frac{2T_\diamond(x^k)}{k}) + \exp(\sum_{k\geq 1} \frac{T_\diamond(x^{2k})}{k}))$. The system $\{y = F(x,y), 1 = F_y(x,y)\}$ can be solved numerically, using truncations of the functions $T_\diamond(x^k)$. Then, one obtains an expansion of the form $A_0 + \sum_{k\geq 1} A_k X^k$ for $y := T_\diamond$. The rest of the functions defined in 4.3 inherit the same type of expansion at the same point ρ and all the involved coefficients are computable.

Using Equation 1, we can obtain a similar expansion for $Z(x)$ (recall that $Z(x) = T(x)$ by Lemma 4.1), only now Z_1 is zero. We can show this formally, by obtaining an expression for Z_1 depending on A_i and then noticing that $0 = F_y(\rho, y_0) - 1 = \frac{1}{A_1} Z_1$. It is easy to see that Z_3 does not vanish, with a combinatorial argument. The result also holds for $G(x)$, using Lemma 4.2. □

The final estimate follows by applying the *Transfer Theorems* of singularity analysis [**12**, Corrollary VI.1, Theorem VI.4].

Corollary 4.5. *The coefficients of $Z(x), G(x)$ satisfy an asymptotic growth of the form*

$$cn^{-\frac{5}{2}}\rho^{-n},$$

where c is equal to $\frac{Z_3}{\Gamma(-3/2)} \approx 0.27160$ and $\frac{G_3}{\Gamma(-3/2)} \approx 0.33995$, respectively, and $\rho^{-1} \approx 6.27888$.

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A. Leivaditis, Department of Mathematics, National and Kapodistrian University of Athens, Athens, Greece,
e-mail: livaditisalex@gmail.com

A. Singh, Inter-university Postgraduate Programme "Algorithms, Logic, and Discrete Mathematics" (ALMA), Athens, Greece,
e-mail: singh.alexandros@gmail.com

G. Stamoulis, Department of Mathematics, National and Kapodistrian University of Athens and Inter-university Postgraduate Programme "Algorithms, Logic, and Discrete Mathematics" (ALMA), Athens, Greece,
e-mail: giannosstam@di.uoa.gr

D. M. Thilikos, AlGCo project-team, LIRMM, Université de Montpellier, CNRS, Montpellier, France,
Department of Mathematics, National and Kapodistrian University of Athens, Inter-university Postgraduate Programme "Algorithms Logic, and Discrete Mathematics", Athens, Greece,
e-mail: sedthilk@thilikos.info

K. Tsatsanis, Inter-university Postgraduate Programme "Algorithms, Logic, and Discrete Mathematics" (ALMA), Athens, Greece,
e-mail: kostistsatsanis@gmail.com

V. Velona, Department of Economics, Universitat Pompeu Fabra, and Departament of Mathematics, Universitat Politècnica de Catalunya, and Barcelona Graduate School of Mathematics, Barcelona, Spain,
e-mail: vasiliki.velona@upf.edu