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RESOLUTION OF THE OBERWOLFACH PROBLEM

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Abstract. The Oberwolfach problem, posed by Ringel in 1967, asks for a decomposition of K_{2n+1} into edge-disjoint copies of a given 2-factor. We show that this can be achieved for all large n. We actually prove a significantly more general result, which allows for decompositions into more general types of factors. In particular, this also resolves the Hamilton-Waterloo problem for large n.

1. Introduction

A central theme in Combinatorics and related areas is the decomposition of large discrete objects into simpler or smaller objects. In graph theory, this can be traced back to the 18th century, when Euler asked for which orders orthogonal Latin squares exist (which was finally answered by Bose, Shrikhande, and Parker [**5**]). This question can be reformulated as the existence question for resolvable triangle decompositions in the balanced complete tripartite graph. (Here a resolvable triangle decomposition is a decomposition into edge-disjoint triangle factors.) In the 19th century, Walecki proved the existence of decompositions of the complete graph K_n (with n odd) into edge-disjoint Hamilton cycles and Kirkman formulated the school girl problem. The latter triggered the question for which n the complete graph on n vertices admits a resolvable triangle decomposition, which was finally resolved in the 1970s by Ray-Chaudhuri and Wilson [**30**] and independently by Jiaxi. This topic has developed into a vast area with connections e.g. to statistical design and scheduling, Latin squares and arrays, graph labellings as well as combinatorial probability.

A far reaching generalisation of Walecki's theorem and Kirkman's school girl problem is the following problem posed by Ringel in Oberwolfach in 1967 (cf. [**25**]).

Problem 1.1 (Oberwolfach problem)**.** Let $n \in \mathbb{N}$ and let F be a 2-regular graph on n vertices. For which (odd) n and F does K_n decompose into edge-disjoint copies of F?

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Addressing conference participants in Oberwolfach, Ringel fittingly formulated his problem as a scheduling assignment for diners: assume n people are to be seated around round tables for $\frac{n-1}{2}$ meals, where the total number of seats is equal to n, but the tables may have different sizes. Is it possible to find a seating chart such that every person sits next to any other person exactly once?

We answer this affirmatively for all sufficiently large n. Note that the Oberwolfach problem does not have a positive solution for every odd n and F (indeed, there are four known exceptions).

A further generalisation is the Hamilton-Waterloo problem; here, two cycle factors are given and it is prescribed how often each of them is to be used in the decomposition. We also resolve this problem in the affirmative (for large n) via the following even more general result. We allow an arbitrary collection of types of cycle factors, as long as one type appears linearly many times. This immediately implies that the Hamilton-Waterloo problem has a solution for large n for any bounded number of given cycle factors.

Theorem 1.2. *For every $\alpha > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all odd $n \geq n_0$ the following holds. Let $F_1, \ldots, F_k$ be 2-regular graphs on n vertices and let $m_1, \ldots, m_k \in \mathbb{N}$ be such that $\sum_{i\in[k]} m_i = (n-1)/2$ and $m_1 \geq \alpha n$. Then K_n admits a decomposition into graphs $H_1, \ldots, H_{(n-1)/2}$ such that for exactly m_i integers j, the graph H_j is isomorphic to F_i.*

Here we say a graph G admits a *decomposition* into graphs $H_1, \ldots, H_t$ if there exist edge-disjoint copies of $H_1, \ldots, H_t$ in G such that every edge of G belongs to exactly one copy.

Several authors (see e.g. Huang, Kotzig, and Rosa [**19**]) considered a variant of the Oberwolfach problem for even n; to be precise, here we ask for a decomposition of K_n minus a perfect matching into $n/2-1$ copies of some given n-vertex 2-regular graph F. We will deduce Theorem 1.2 from a more general result (Theorem 1.3) which also covers this case.

The Oberwolfach problem and its variants have attracted the attention of many researchers, resulting in more than 100 research papers covering a large number of partial results. Most notably, Bryant and Scharaschkin [**8**] proved it for infinitely many n. Traetta [**31**] solved the case when F consists of two cycles only, Bryant and Danziger [**6**] solved the variant for even n if all cycles are of even length, Alspach, Schellenberg, Stinson, and Wagner [**3**] solved the case when all cycles have equal length (see [**18**] for the analogous result for n even), and Hilton and Johnson [**17**] solved the case when all but one cycle have equal length.

A related conjecture of Alspach stated that for all odd n the complete graph K_n can be decomposed into any collection of cycles of length at most n whose lengths sum up to $\binom{n}{2}$. This was solved by Bryant, Horsley, and Pettersson [**7**]. Also similar in spirit to the Oberwolfach problem, Ringel conjectured that for every n and every tree T on $n+1$ vertices, $2n+1$ copies of T decompose K_{2n+1}. This has been solved if T has bounded maximum degree in [**20**], and solved approximately (for slightly smaller trees) by Montgomery, Pokrovskiy, and Sudakov [**27**].

Most classical results in the area are based on algebraic approaches, often by exploiting symmetries. More recently, major progress for decomposition problems has been achieved via absorbing techniques in combination with approximate decomposition results (often also in conjunction with probabilistic ideas). This started off with decompositions into Hamilton cycles [**10, 24**], followed by the existence of combinatorial designs [**15, 16, 21, 22**] and progress on the tree packing conjecture [**20**]. In this paper, at a very high level, we also pursue such an approach. As approximate decomposition results, we exploit a hypergraph matching argument due to Alon and Yuster [**2**] (which in turn is based on the Rödl nibble via the Pippenger-Spencer theorem [**29**]) and a bandwidth theorem for approximate decompositions due to Condon, Kim, Kühn, and Osthus [**9**]. Our absorption procedure utilizes as a key element a very special case of a recent result of Keevash on resolvable designs [**22**].

Whenever we only seek an approximate decomposition of a graph G, the target graphs can be significantly more general and divisibility conditions disappear. In particular, Allen, Böttcher, Hladký, and Piguet [**1**] considered approximate decompositions into graphs of bounded degeneracy and maximum degree $o(n/\log n)$ whenever the host graph G is sufficiently quasirandom, and Kim, Kühn, Osthus, and Tyomkyn [**23**] considered approximate decompositions into graphs of bounded degree in host graphs G satisfying weaker quasirandom properties (namely, ε-superregularity). Their resulting blow-up lemma for approximate decompositions was a key ingredient for [**9, 20**] (and thus for the current paper too). It also implies that an approximate solution to the Oberwolfach problem can always be found (this was obtained independently by Ferber, Lee, and Mousset [**12**]).

Our Theorem 1.2 actually follows from the following more general Theorem 1.3, which allows separable graphs. An n-vertex graph H is said to be *ξ-separable* if there exists a set S of at most ξn vertices such that every component of $H \setminus S$ has size at most ξn. Examples of separable graphs include cycles, powers of cycles, planar graphs, and F-factors. More generally, for bounded degree graphs, the notion of separability is equivalent to that of small bandwidth.

Theorem 1.3. *For given $\Delta \in \mathbb{N}$ and $\alpha > 0$, there exist $\xi_0 > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and $\xi < \xi_0$. Let $\mathcal{F}, \mathcal{H}$ be collections of graphs satisfying the following:*

- *$\mathcal{F}$ is a collection of at least αn copies of F, where F is a 2-regular n-vertex graph;*
- *each $H \in \mathcal{H}$ is a ξ-separable n-vertex r_H-regular graph for some $r_H \leq \Delta$;*
- *$e(\mathcal{F} \cup \mathcal{H}) = \binom{n}{2}$.*

Then K_n decomposes into $\mathcal{F} \cup \mathcal{H}$.

Clearly, Theorem 1.3 implies Theorem 1.2 and also its corresponding version if n is even and we ask for a decomposition of K_n minus a perfect matching.

While far more general than the Oberwolfach problem, Theorem 1.3 may be just the tip of the iceberg, and it seems possible that the following is true.

Conjecture 1.4. For all $\Delta \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ so that the following holds for all $n \geq n_0$. Let $F_1, \ldots, F_t$ be n-vertex graphs such that F_i is r_i-regular for some $r_i \leq \Delta$ and $\sum_{i \in [t]} r_i = n-1$. Then there is a decomposition of K_n into $F_1, \ldots, F_t$.

The above conjecture is implicit in the 'meta-conjecture' on decompositions proposed in [**1**].

Rather than considering decompositions of the complete graph K_n, it is also natural to consider decompositions of host graphs of large minimum degree (this has applications e.g. to the completion of partial decompositions of K_n). Indeed, a famous conjecture of Nash-Williams [**28**] states that every n-vertex graph G of minimum degree at least $3n/4$ has a triangle decomposition (subject to the necessary divisibility conditions). The following conjecture would (asymptotically) transfer this to arbitrary 2-regular spanning graphs.

Conjecture 1.5. Suppose G is an n-vertex r-regular graph with even $r \geq \frac{3}{4}n + o(n)$ and F is a 2-regular graph on n vertices. Then G decomposes into copies of F.

The (asymptotic version of the) Nash-Williams conjecture was reduced to its fractional version in [**4**]. In combination with [**11**], this shows that the Nash-Williams conjecture holds with $3n/4$ replaced by $9n/10 + o(n)$. There has also been considerable progress on decomposition problems involving such host graphs of large minimum degree into other fixed subgraphs H rather than triangles [**14, 26**]. It turns out that the chromatic number of H is a crucial parameter for this problem. In particular, as proved in [**14**], for bipartite graphs H the 'decomposition threshold' is always at most $\frac{2}{3}n + o(n)$.

Clearly, one can generalise Conjecture 1.5 in this direction, e.g. to determine the decomposition threshold for K_r-factors. It might also be true that the '3/4' in Conjecture 1.5 can be replaced by '2/3' if F consists only of even cycles. We are confident that the ideas from this paper will be helpful in approaching these and other related problems. A full version of this paper can be found here [**13**].

2. Proof sketch

For simplicity, we just sketch the argument for the setting of the Oberwolfach problem; that is, we aim to decompose K_n into $\frac{n-1}{2}$ copies of an n-vertex 2-regular graph F. The proof essentially splits into two cases. In the first case we assume that almost all vertices of F belong to 'short' cycles, of length at most 500. Note that there must be some cycle length, say ℓ^*, such that at least $n/600$ vertices of F lie in cycles of length ℓ^*. We will take a suitable number of random slices of the edges of K_n and then first embed, for every desired copy of F, all cycles whose lengths are different from ℓ^*. For this, we use standard tools based on the Rödl nibble. We then complete the decomposition by embedding all the cycles of length ℓ^*. This last step uses a special case of a recent result of Keevash on the existence of resolvable designs [**22**].

The second case is much more involved and forms the core of the proof. We are now guaranteed that a (small) proportion of vertices of F lies in 'long' cycles. To motivate our approach, consider the following simplified setup. Suppose F consists only of cycles whose lengths are divisible by 3, and suppose for the moment we seek an F-decomposition of a 3-partite graph G with equitable vertex partition (V_1, V_2, V_3) (so G is a C_3-blowup). Let $\ell_1, \ldots, \ell_t$ be the sequence of cycle lengths appearing in F. Now, take any permutation π on V_3 which consists of cycles of lengths $\ell_1/3, \ldots, \ell_t/3$. For instance, a C_3 in F corresponds to a fixed point in π, and a C_6 in F corresponds to a transposition in π. Now, define an auxiliary graph $\pi(G)$ by 'rewiring' the edges between V_2 and V_3 according to π. More precisely, we ensure that $E_{\pi(G)}(V_2, V_3) = \{v_2\pi(v_3) : v_2v_3 \in E(G)\}$. Suppose that F' is a C_3-factor in $\pi(G)$. By 'reversing' the rewiring, we obtain a copy of F in G. More precisely, let $\pi^{-1}(F')$ be the graph obtained from F' by replacing $F'[V_2, V_3]$ with $\{v_2\pi^{-1}(v_3) : v_2v_3 \in E(F')\}$. Clearly, $\pi^{-1}(F') \cong F$ and $\pi^{-1}(F') \subseteq G$. What is more, this rewiring is canonical in the following sense: if F' and F'' are edge-disjoint C_3-factors in $\pi(G)$, then $\pi^{-1}(F')$ and $\pi^{-1}(F'')$ will be edge-disjoint copies of F in G. Thus, a resolvable C_3-decomposition of $\pi(G)$ immediately translates into an F-decomposition of G.

Similarly, if all cycle lengths in F are divisible by 4, we can reduce the problem of finding an F-decomposition of a C_4-blowup to the problem of finding a resolvable C_4-decomposition of a suitably rewired C_4-blowup. In order to deal with arbitrary 2-regular graphs F, we interweave such constructions for C_3, C_4 and C_5. In our proof we construct an 'absorbing graph' G which is a partite graph on 18 vertex classes such that finding an F-decomposition of G can be reduced to finding resolvable C_3, C_4, C_5-decompositions of suitable auxiliary graphs, in a similar way as sketched above. Crucially, G has this property in a robust sense: even if we delete an arbitrary sparse graph L from G, as long as some necessary divisibility conditions hold, we are still able to find an F-decomposition of $G - L$.

The overall strategy is thus as follows: first, we remove G from K_n. Then we find an approximate decomposition of the remainder, which leaves a sparse leftover. For this, we employ the recent bandwidth theorem for approximate decompositions [**9**]. (The existence of an approximate decomposition of $K_n - G$ would also follow directly from the blow-up lemma for approximate decompositions [**23**], but this would leave a leftover whose density is larger than that of the absorbing graph G, making our approach infeasible.) We then deal with this leftover by using some edges of G, in a very careful way, such that the remainder of G is still appropriately divisible. The remainder of G then decomposes as sketched above. In order to decompose the auxiliary graphs, we again use a very special case of the main result in [**22**]. The fact that we are guaranteed that F has some long cycles will be helpful to construct the absorbing graph G, more precisely, to ensure that all the 18 vertex classes are of linear size. It is also essential when dealing with the leftover of the approximate decomposition.

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