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ON 2-FACTORS WITH A SPECIFIED NUMBER OF COMPONENTS IN LINE GRAPHS

S. CHIBA, Y. EGAWA, J. FUJISAWA, A. SAITO, I. SCHIERMEYER, M. TSUGAKI and T. YAMASHITA

Abstract. Kaiser and Vrána [European J. Combin. **33** (2012), 924–947] showed that every 5-connected line graph of minimum degree at least 6 is hamiltonian, which gives a partial solution to Thomassen's Conjecture on hamiltonicity of line graphs [J. Graph Theory **10** (1986), 309–324]. In this paper, we prove that every 5-connected line graph of sufficiently large order compared with a given positive integer k and of minimum degree at least 6 also has a 2-factor with exactly k cycles. In order to show this result, we investigate minimum degree conditions for the existence of such a 2-factor in hamiltonian line graphs.

1. Introduction

In this paper, a *graph* or a *simple graph* means a finite undirected graph without loops or multiple edges. A *multigraph* may contain multiple edges but no loops. For a (multi)graph G, we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of G, respectively. The *line graph* of a (multi)graph H, denoted by $L(H)$, is the graph on vertex set $E(H)$ in which two vertices in $L(H)$ are adjacent if and only if their corresponding edges in H share an end vertex. A graph G is a *line graph* if it is isomorphic to $L(H)$ for some (multi)graph H, and we call H a *preimage* of G.

A graph is said to be *hamiltonian* if it has a *Hamilton cycle*, i.e., a cycle containing all the vertices. The problem of finding a Hamilton cycle is a fundamental problem in graph theory. But, it is NP-complete, and so various kinds of sufficient conditions for hamiltonicity of graphs have been extensively studied (see [**6, 7**]). In the literature, there also remain important unsolved problems. For instance, Thomassen posed the following to approach Chvátal's Toughness Conjecture ("every 2-tough graph is hamiltonian").

Conjecture A (Thomassen [**15**])**.** Every 4-connected line graph is hamiltonian.

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It is known that Conjecture A is equivalent to several other statements, some seemingly weaker or stronger than Conjecture A. For example, Ryjáček [**13**] proved that Conjecture A is equivalent to Matthews and Sumner's Conjecture [**11**] which states that every 4-connected claw-free graph is hamiltonian. The Dominating Cycle Conjecture ("every cyclically 4-edge-connected cubic graph has a dominating cycle") by Fleischner is also known to be equivalent to Conjecture A, see the paper by Fleischner and Jackson [**5**]. For other results and conjectures, we refer the reader to [**2**].

On the other hand, as a positive result related to Conjecture A, Zhan [**16**] and independently Jackson (unpublished) proved that every 7-connected line graph is hamiltonian. Kaiser and Vrána, in 2012, improved this result as follows, which is the currently best connectivity result related to Conjecture A.

Theorem B (Kaiser, Vrána [**10**]). *Every 5-connected line graph G of a multigraph with $\delta(G) \geq 6$ is hamiltonian.*

A 2-*factor* of a graph is a spanning subgraph in which every component is a cycle. Thus a Hamilton cycle is a 2-factor with "exactly 1 component (cycle)". This naturally leads us to consider sufficient conditions for a graph to have a 2-factor with a specified number of components. In fact, Brandt et al. [**1**] studied this problem and showed that the same degree condition as the classical Ore's Theorem [**12**], i.e., the degree sum is at least $|V(G)|$ for any pair of non-adjacent vertices in a graph G, also guarantees the existence of such a 2-factor for sufficiently large graphs. Related results are also shown for dense graphs (see e.g., [**4**]).

Motivated by Conjecture A and Theorem B, in this research, we consider the above problem for line graphs. The following result, which is our main result, shows that the same degree condition as in Theorem B also guarantees the existence of a 2-factor with exactly k (≥ 2) cycles for sufficiently large hamiltonian line graphs.

Theorem 1. *For every integer k with $k \geq 2$, there exists an integer $n = n(k)$ such that every hamiltonian line graph G of a multigraph with $|V(G)| \geq n$ and $\delta(G) \geq 6$ has a 2-factor with exactly k cycles.*

Our proof gives $n = O(k^6)$. We do not believe that it is the optimum order for n. But, for the sake of clarity in the proof, we do not attempt to improve n. (We upload the full version of the paper at our website [**3**]. See it for the detail of the proof.)

By Theorems 1 and B, we get the following.

Corollary 2. *For every positive integer k, there exists an integer $n = n(k)$ such that every 5-connected line graph G of a multigraph with $|V(G)| \geq n$ and $\delta(G) \geq 6$ has a 2-factor with exactly k cycles.*

In order to prove Theorem 1, we consider decompositions of a "large circuit" into a prescribed number of edge-disjoint circuits. In the next section, we introduce the key lemma (Lemma 2) for the proof of Theorem 1. In Section 3, we give some remarks on the minimum degree condition in Theorem 1 and some open problems.

2. Circuit decompositions

Gould and Hynds [8] introduced the notion of a k-system that dominates a given simple graph, and gave a criterion for a line graph of a simple graph to have a 2-factor with exactly k cycles. In this paper, we first extend this notion so that it works in the class of line graphs of multigraphs (see Lemma 1). Note that a circuit in a graph H naturally corresponds to a cycle in $L(H)$ in the class of simple graphs. But in the class of multigraphs, a circuit of length 2 may arise, which does not correspond to a cycle in the line graph.

A *circuit* is a connected multigraph in which every vertex has a positive even degree. A *star* is a multigraph consisting of at least two edges incident with one common vertex. Hence, in this paper, a star may contain a circuit. Note also that there is even a graph which is both a star and a circuit. Now let k be a positive integer, and let H be a multigraph and $\mathcal{S}$ be a set of circuits and stars in H. If $\mathcal{S}$ satisfies the following conditions (S1)–(S4), then $\mathcal{S}$ is called a *k-system that dominates H*. Here, for a set $\mathcal{F}$ of graphs, we define $E(\mathcal{F}) = \bigcup_{F\in\mathcal{F}} E(F)$.

(S1) $|\mathcal{S}| = k$.
(S2) Any two distinct elements of $\mathcal{S}$ are edge-disjoint in H.
(S3) Every element of $\mathcal{S}$ has size at least 3.
(S4) Every edge in $E(H) \setminus E(\mathcal{S})$ is incident with a circuit of an element of $\mathcal{S}$.

By using a theorem of Harary and Nash-Williams [9] concerning a necessary and sufficient condition for the existence of a Hamilton cycle in line graphs, we can show the following.

Lemma 1. *Let k be a positive integer, and let H be a multigraph. Then the line graph $L(H)$ has a 2-factor with exactly k cycles if and only if H has a k-system that dominates H.*

Therefore it suffices to show the existence of a k-system that dominates a preimage H of a line graph G in the proof of Theorem 1.

Let H be a multigraph. For two vertices x and y, the number of edges of H joining them is called the *edge-multiplicity* of x and y. The maximum edge-multiplicity over all pairs of vertices is denoted by $\mu(H)$. For an integer $s \geq 2$, an *s-bond* of H is a set of s edges joining two vertices of edge-multiplicity exactly s. In particular, an s-bond is said to be *pendant* if one of the two ends, say x, satisfies $\deg_H(x) = s$, where $\deg_H(x)$ denotes the degree of x in H. We also define $V_{\geq 4}(H) = \{v \in V(H) : \deg_H(v) \geq 4\}$. The following lemma plays a crucial role to construct a k-system in the proof of Theorem 1.

Lemma 2. *Let l be an integer with $l \geq 2$, and let C be a circuit. Suppose that C satisfies the following conditions* (a-1)–(a-3)*:*

(a-1) $\mu(C) \leq 2$.
(a-2) *C does not have a pendant 2-bond.*
(a-3) $$\sum_{v\in V_{\geq 4}(C)} \deg_C(v) \geq 4\left(1 + \frac{(l-2)(4l^2+5l+9)}{3}\right).$$

Then C contains l edge-disjoint circuits $C_1, \ldots, C_l$ such that $|E(C_p)| \geq 3$ for $1 \leq p \leq l$ and $\bigcup_{1 \leq p \leq l} E(C_p) = E(C)$.

We now give a high-level overview of the proof of Theorem 1: Let H be a multigraph such that $L(H)$ is a hamiltonian graph of sufficiently large order and $\delta(L(H)) \geq 6$. Then H has a dominating circuit C, i.e., each edge of H is incident to a vertex of C. If the degree sum of vertices with degree at least 4 in C is large compared with k, then by removing the appropriate number of edges from s-bonds ($s \geq 3$) and pendant 2-bonds of C and, if possible, by taking edge-disjoint stars of size 3 or 4 consisting of the removed edges and edges in $E(H) \setminus E(C)$ as many as possible, we construct a circuit $\tilde{C}$ from C so that $\tilde{C}$ satisfies (a-1)–(a-3) in Lemma 2 for some integer l depending on k and the number of the stars of size 3 or 4 (note that we use the degree condition in this argument). Then we can construct a k-system that dominates H from the circuit decomposition of $\tilde{C}$ and the stars. On the other hand, for the case where the degree sum is small, we take edge-disjoint stars of size 3 consisting of edges in $E(H) \setminus E(C)$ as many as possible, and then we can construct a k-system that dominates H from C and the stars; otherwise, we can get a larger dominating circuit and this results in the first case (note that we also use the degree condition in this argument). We refer to [**3**] for the detail.

3. Future work

In this research, we have investigated sufficient conditions for the existence of a 2-factor with a specified number of components in line graphs, and have shown that for a sufficiently large hamiltonian line graph, minimum degree at least 6 suffices for the existence of such a 2-factor (Theorem 1). However, we do not know whether the lower bound on the minimum degree condition in Theorem 1 is sharp or not. The following example shows that minimum degree at least 5 is necessary: Let k be an integer with $k \geq 2$, and let n be a sufficiently large integer compared with k. Let H be the multigraph obtained from a cycle $u_1 v_1 u_2 v_2 \ldots u_n v_n u_1$ of order $2n$ by replacing the edge $u_i v_i$ with a 3-bond for $1 \leq i \leq n$. Then $L(H)$ is a hamiltonian graph of minimum degree 4. Let $I = \{v_i u_{i+1} \mid 1 \leq i \leq n\}$, where $u_{n+1} = u_1$. Then, for any cycle C in $L(H)$, $|V(C) \cap I| = 1$, 2 or n. Since $n \gg k$, this yields that $L(H)$ does not have a 2-factor with exactly k cycles. Hence we can raise the following problem.

Problem 1. Is it true that, for every integer k with $k \geq 2$, there exists an integer $n = n(k)$ such that every hamiltonian line graph G of a multigraph with $|V(G)| \geq n$ and $\delta(G) \geq 5$ has a 2-factor with exactly k cycles?

As a possible extension of the research in this paper, we also raise the problem on the existence of a 2-factor with a specified number of components in a claw-free graph (i.e., a graph which contains no induced $K_{1,3}$). For problems on hamiltonicity of claw-free graphs, the closure concept introduced by Ryjáček [**13**] can be a very useful tool. In fact, there are many results which rely on the Ryjáček closure, i.e., the results are shown for line graphs first, and then these are deduced by stability in the class of claw-free graphs under the closure. Similar to the relationship between Thomassen's Conjecture and Matthews-Sumner's Conjecture, Theorem B was extended to claw-free graphs by using the closure concept (see [**10**, Corollary 25]). However, having a 2-factor with exactly k cycles is not stable in the class of claw-free graphs under the Ryjáček closure (see [**14**]). Therefore, we need another approach to extend Theorem 1 to claw-free graphs, and considering it might be interesting.

Problem 2. Determine a sharp minimum degree condition to guarantee that a hamiltonian claw-free graph (of sufficiently large order) has a 2-factor with exactly k cycles. Is minimum degree at least 5 enough?

References

1. Brandt S., Chen G., Faudree R., Gould R. J. and Lesniak L., *Degree conditions for 2-factors*, J. Graph Theory **24** (1997), 165–173.

2. Broersma H. J., Ryjáček Z. and Vrána P., *How many conjectures can you stand? A survey*, Graphs Combin. **28** (2012), 57–75.

3. Chiba S., Egawa Y., Fujisawa J., Saito A., Schiermeyer I., Tsugaki M. and Yamashita T., `https://drive.google.com/file/d/137EnO8gAuYL-J62Fuo-FXGkZ3tgQvDHx/view?usp=sharing`

4. DeBiasio L., Ferrara M. and Morris T., *Improved degree conditions for 2-factors with k cycles in hamiltonian graphs*, Discrete Math. **320** (2014), 51–54.

5. Fleischner H. and Jackson B., *A note concerning some conjectures on cyclically 4-edge-connected 3-regular graphs*, in: Graph Theory in Memory of G.A. Dirac, Annals of Discrete Mathematics 41, North-Holland, Amsterdam, 1989, 171–177.

6. Gould R. J., *Advances on the Hamiltonian problem – a survey*, Graphs Combin. **19** (2003), 7–52.

7. Gould R. J., *Recent advances on the Hamiltonian problem: Survey III*, Graphs Combin. **30** (2014), 1–46.

8. Gould R. J. and Hynds E. A., *A note on cycles in 2-factors of line graphs*, Bull. Inst. Combin. Appl. **26** (1999), 46–48.

9. Harary F. and Nash-Williams C. St. J. A., *On eulerian and hamiltonian graphs and line graphs*, Canad. Math. Bull. B **8** (1965), 701–709.

10. Kaiser T. and Vrána P., *Hamilton cycles in 5-connected line graphs*, European J. Combin. **33** (2012), 924–947.

11. Matthews M. M. and Sumner D. P., *Hamiltonian results in $K_{1,3}$-free graphs*, J. Graph Theory **8** (1984), 139–146.

12. Ore O., *Note on hamilton circuits*, Amer. Math. Monthly **67** (1960), 55.

13. Ryjáček Z., *On a closure concept in claw-free graphs*, J. Combin. Theory Ser. B **70** (1997), 217–224.

14. Ryjáček Z., Saito A. and Schelp R. H., *Closure, 2-factors, and cycle coverings in claw-free graphs*, J. Graph Theory **32** (1999), 109–117.

15. Thomassen C., *Reflections on graph theory*, J. Graph Theory **10** (1986), 309–324.

16. Zhan S., *On Hamiltonian line graphs and connectivity*, Discrete Math. **89** (1991), 89–95.

S. Chiba, Applied Mathematics, Faculty of Advanced Science and Technology, Kumamoto University, Kumamoto, Japan,
e-mail: schiba@kumamoto-u.ac.jp

Y. Egawa, Department of Applied Mathematics, Tokyo University of Science, Tokyo, Japan

J. Fujisawa, Faculty of Business and Commerce, Keio University, Yokohama, Japan,
e-mail: fujisawa@fbc.keio.ac.jp

A. Saito, Department of Information Science, Nihon University, Tokyo, Japan,
e-mail: asaito@chs.nihon-u.ac.jp

I. Schiermeyer, Institut für Diskrete Mathematik und Algebra, Technische Universität Bergakademie Freiberg, Freiberg, Germany,
e-mail: Ingo.Schiermeyer@tu-freiberg.de

M. Tsugaki, Tokyo University of Science, Tokyo, Japan,
e-mail: tsugaki@hotmail.com

T. Yamashita, Department of Mathematics, Kindai University, Osaka, Japan,
e-mail: yamashita@math.kindai.ac.jp