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MB-HOMOGENEOUS GRAPHS AND SOME NEW HH-HOMOGENEOUS GRAPHS

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Abstract. We present a result showing that any countably infinite HH-homogeneous graph that does not contain the Rado graph as a spanning subgraph has finite independence number; from this we derive a classification of MB-homogeneous graphs. Additionally, we present constructions that yield new HH-homogeneous graphs.

Homomorphism-homogeneity was introduced in [**3**] as a variation on ultrahomogeneity. A relational structure G is homomorphism-homogeneous if every homomorphism f between finite induced substructures is the restriction of an endomorphism F of G to the domain of f. This definition can be refined by specifying what kind of homomorphism is f and what kind of endomorphism F is; Lockett and Truss introduced many of these classes in [**6**]. Following the tradition, we specify a morphism-extension class by two characters XY, where X comes from $\{H, M, I\}$ (the letters stand for Homomorphism, Monomorphism, and Isomorphism) and Y comes from $\{H, I, A, B, M\}$ (corresponding to endomorphism, Isomorphism, Automorphism, Bimorphism, Monomorphism). Then G is XY-homogeneous if every X-morphism between finite substructures extends to a global Y-morphism.

Attempts at classification followed the original Cameron-Nešetřil paper. We have two levels of classification: first, the basic question of what the partial order of morphism-extension classes looks like for a given class of structures (Rusinov and Schweitzer proved the equality of HH and MH for graphs and separated some other classes in [**7**], and other authors have studied similar problems for the relatively general L-colored graphs, finite [**5**] and countably infinite [**1**]). A more ambitious project is to classify each class. For example, classify all countable HH-homogeneous graphs. Although classification of finite undirected HH-homogeneous graphs is easy and finished [**3**], the same task for countably inifinite graphs remains open. Any classification will necessarily be up to a much coarser equivalence relation than isomorphism, because there are uncountably many non-isomorphic HH-homogeneous graphs.

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We provide a classification of countably infinite MB-homogeneous graphs up to bimorphism-equivalence, answering a question from [**4**], as well as some preliminary results for a classification of HH-homogeneous graphs. Full versions of the arguments that prove the classification of MB-homogeneous graphs can be found in [**2**]. All graphs in this paper are countably infinite.

Let $(\triangle)$ denote the property of containing the Rado graph as a spanning subgraph. In the original paper [**3**], it was observed that any countably infinite graph that satisfies $(\triangle)$ is homomorphism-homogeneous, but not all HH-homogeneous graphs satisfy $(\triangle)$, as was shown by Rusinov and Schweitzer in [**7**]. Until recently, their examples and unions of equipotent complete graphs were the only known families of HH-homogeneous graphs with $\neg(\triangle)$.

We don't know enough about HH-homogeneous graphs to say it more than tentatively, but the subclass of HH-homogeneous graphs that do not satisfy $(\triangle)$ may be amenable to classification. In an attempt to eliminate potentially problematic cases, we decided to test whether an infinite connected graph with $\neg(\triangle)$ can embed arbitrarily large independent subsets.

It is an easy observation that if G is a countable HH-homogeneous graph that does not satisfy $(\triangle)$, then the number $o(G) = \sup\{\alpha(N(v)) : v \in G\}$ is finite. As usual, $\alpha(G)$ denotes the independence number of G, or $\sup\{|X| : X$ is an independent subset of $G\}$. We don't know a better short name for $o(G)$, so we call it the *codependence number* of G.

An obvious way to produce a countably infinite HH-homogeneous graph with finite codependence number is to impose a finite bound on the independence number, since the independence number is an upper bound on the codependence number. The Rusinov-Schweitzer examples (see [**7**]) do exactly that. But is it possible to construct a countably infinite connected graph that does not satisfy $(\triangle)$ and has infinite independence number? The answer is no.

Theorem 1. *Let G be an infinite connected HH-homogeneous graph, and suppose $o(G) \in \mathbb{N}$. Then $\alpha(G) < 2o(G) + \left\lceil \frac{o(G)}{2} \right\rceil - 1$.*

We prove the theorem by contradiction. If G is a graph with sufficiently large independence number and I is a maximal independent subset witnessing it, then by maximality of I, the neighbourhood of any vertex outside I contains elements of I. We call $N(v) \cap I$ the *address* of v (if $v \in I$, then $\{v\}$ is its address).

Each subset S of I of size $o(G)$ defines an infinite HH-homogeneous subgraph of G with finite independence number, namely the graph induced on $R = \bigcap_{s \in S} N(s)$, and we can prove under these conditions that the structure of the induced subgraph of G whose vertex set is the union of I and the elements with address of size $o(G)$ is closely related to the intersection graph of all subsets of I with $o(G)$ elements. In particular, R contains two types of triangles, which we use to find two isomorphic finite subgraphs X and Y such that G contains a vertex x whose neighbourhood contains X, but for any $y \in G$, the neighbourhood of y does not contain Y. This contradicts HH-homogeneity, since any isomorphism $X \to Y$ is a homomorphism, but there is no way to extend it to x.

Now we can use Theorem 1 to classify MB-homogeneous graphs.

The surjectivity of a global extension of a finite monomorphism allows us to pull back non-edges over the image of a finite homomorphism. More formally, if X is a finite subset of an MB-homogeneous graph and $c \not\sim x$ for some $c \notin X$, then if $f\colon Y \to X$ is any monomorphism, then we know that the preimage of c under a global extension of f will satisfy $F^{-1}(x) \not\sim F^{-1}(c)$. We use this technique to prove that if X is a finite subset of G and X has a co-cone (that is, if there exists a vertex $v \notin X$ such that $v \not\sim x$ for each $x \in X$), then there exists an infinite independent subset of G consisting of co-cones over X. Therefore, any non-complete MB-homogeneous graph has infinite independence number, and by Theorem 1 satisfies ($\triangle$).

A countable graph G is bimorphism-equivalent to the Rado graph iff every finite subset has a cone and a co-cone in G. A result from [**4**] states that the class of MB-homogeneous graphs is closed under complements, so if both G and $\bar{G}$ are connected, then we get bimorphism-equivalence to the Rado graph. Thus, we concentrate on disconnected MB-homogeneous graphs, omitting the obvious $\overline{K_\omega}$.

Lemma 2. *If G is a non-complete countably infinite connected MB-homogeneous in which some finite subset has no co-cones, then it is isomorphic to $\overline{\omega \times K_\omega}$.*

This suffices to classify MB-homogeneous graphs up to bimorphism-equivalence.

Theorem 3. *Let G be a countable MB-homogeneous graph. Then G is bimorphism-equivalent to one of the following or its complement:*

1. K_ω,
2. $\omega \times K_\omega$,
3. *The Rado graph $\mathcal{R}$.*

We originally studied the independence number of HH-homogeneous graphs because we were trying to classify the countably infinite connected HH-homogeneous graphs that do not satisfy ($\triangle$). A persistent annoyance is the relative dearth of examples. The only HH-homogeneous graphs in the literature are unions of cliques, graphs with ($\triangle$), and the Rusinov-Schweitzer examples, so testing hypotheses is slow. This list of examples is heavily skewed towards ($\triangle$) and a positive but minimal $\alpha(G) - o(G)$. A simple modification of the Rusinov-Schweitzer construction allows us to produce new graphs where the difference $\alpha(G) - o(G)$ is larger.

The original construction for a graph with $\neg(\triangle)$ is as follows: leg I_n be an independent set on $n \geq 3$ vertices. For each subset S of I of size $n-1$, add an infinite clique K_S of new vertices. The vertices of K_S contain S and $K_{S'}$ in their neighbourhood, for any $S' \neq S$. A shorter description is: given $n \geq 3$, let G_n be the complement of the disjoint union of n copies of $K_{1,\omega}$. A simple modification of the Rusinov-Schweitzer construction (in which we consider subsets of I of size $o < |I|$) tells us that we can push the codependence number down at least to $\lceil \frac{\alpha}{2} \rceil$.

Proposition 4. *For any $o \geq 2$ and α with $o < \alpha < 2o$, the graph* $\mathrm{RS}(\alpha, o)$ *is HH-homogeneous. On the other hand,* $\mathrm{RS}(2o, o)$ *is not HH-homogeneous.*

Therefore we have for each pair (α, o) that satisfies $o < \alpha < 2o$ an example of a connected HH-homogeneous graph with $\neg(\triangle)$, independence number α and codependence number o.

The next question is whether we can produce connected HH-homogeneous graphs with a given finite codependence number and such that some minimal finite induced subgraph without a cone is not an independent set. This is also possible, and yields many examples with $\alpha(G) = o(G)$; no such examples were known before. For a given graph Z, we will denote the result of carrying out the Rusinov-Schweitzer construction starting from Z instead of an independent set by $\mathrm{RS}(Z)$. We refer to any finite induced subgraph F without a cone for which any proper induced subgraph has a cone as a centre of G, even when G is not HH-homogeneous. In $\mathrm{RS}(Z)$, Z is a centre.

In an HH-homogeneous graph with vertices of infinite degree (this is in particular true of any infinite connected HH-homogeneous graph), if c is a cone over H, then there exists a cone over $H \cup \{c\}$. This observation implies in particular that no centre Z of G can contain a vertex of degree $|Z| - 1$. If we lower the maximum degree $\Delta(Z)$ a little bit more, then we can produce HH-homogeneous graphs where Z is a centre.

Proposition 5. *Suppose that Z is a finite graph on $n > 3$ vertices with $\Delta(Z) < n - 2$. Then $\mathrm{RS}(Z)$ is an HH-homogeneous graph with $\alpha(\mathrm{RS}(Z)) = o(\mathrm{RS}(Z)) = \alpha(Z)$.*

We know sufficient conditions that and a "linking" operation that allow us to produce a HH-homogeneous graph for which the set of induced subgraphs that do not have a cone is the set of subgraphs containing (homomorphic preimages of) a graph from a finite list, but such families are hard to find and at this point we have not produced any example with more than 2 non-isomorphic centres. Let us call a set of finite graphs $\{Z_i : i \in I\}$ *admissible* if:

1. For any $i, j \in I$ and proper $P \subset Z_i$, there is no surjective homomorphism $P \to Z_j$.
2. There exists $N \in \mathbb{N}$ such that $\alpha(Z_i) \leq N$ for all $i \in I$
3. For all $i \in I$, $\Delta(Z_i) \leq |Z_i| - 2$

These are all necessary conditions for $\{Z_i : i \in \omega\}$ to be a set of centers of a HH-homogeneous graph. At the time of this writing, we have not succeeded in constructing an HH-homogeneous graph with a given infinite family of centres. We conjecture:

Conjecture 6. Let $(G_i)_{i\in\omega}$ be an admissible family with infinitely many pairwise non-isomorphic finite graphs. There is no HH-homogeneous graph with $\neg(\triangle)$ where the G_i can be embedded as pairwise disjoint centres.

If this conjecture is true, then we can focus on the following problem.

Problem 7. Find an example of a connected HH-homogeneous graph with infinitely many pairwise non-isomorphic centres with nonempty intersection, or prove that no such graph exists.

If no such graphs exist, then the HH-homogeneous graphs with $\neg(\triangle)$ may be classified (up to some equivalence relation, much weaker than isomorphism) by their independence number, their codependence number, and a finite family of finite graphs, namely its minimal (in the partial order by surjective homomorphism) centres. At the time of this writing, it is not clear that this will be the case, though we have an argument that almost proves Conjecture 6. Even if classification is not possible, any advance in this direction will yield interesting graphs.

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