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A RAINBOW BLOW-UP LEMMA FOR ALMOST OPTIMALLY BOUNDED EDGE-COLOURINGS

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Abstract. A subgraph of an edge-coloured graph is called rainbow if all its edges have different colours. We prove a rainbow version of the blow-up lemma of Komlós, Sárközy and Szemerédi that applies for almost optimally bounded edge-colourings. A corollary of this is that there exists a rainbow copy of any bounded-degree spanning subgraph H in a quasirandom host graph G, assuming that the edge-colouring of G fulfills a boundedness condition that can be seen to be almost best possible.

This has many interesting applications beyond rainbow colourings, for example to graph decompositions. There are several well-known conjectures in graph theory concerning tree decompositions, such as Kotzig's conjecture and Ringel's conjecture. We adapt these conjectures to general bounded-degree subgraphs, and provide asymptotic solutions using our result on rainbow embeddings.

1. Introduction

We study rainbow embeddings of bounded-degree spanning subgraphs into quasirandom graphs with almost optimally bounded edge-colourings. Moreover, following the recent work of Montgomery, Pokrovskiy and Sudakov [**28**] on embedding rainbow trees, our main result yields several applications to graph decompositions, graph labellings and orthogonal double covers.

Given a (not necessarily) proper edge-colouring of a graph, a subgraph is called *rainbow* if all its edges have different colours. Rainbow colourings appear in many different contexts of combinatorics, and many problems beyond graph colouring can be translated into a rainbow subgraph problem. What makes this concept so versatile is that it can be used to find 'conflict-free' subgraphs. More precisely, an edge-colouring of a graph G can be interpreted as a system of conflicts on $E(G)$, where two edges conflict if they have the same colour. A subgraph is then conflict-free if and only if it is rainbow.

For instance, rainbow matchings in $K_{n,n}$ can be used to model transversals in Latin squares. The study of Latin squares dates back to the work of Euler in the 18th century and has since been a fascinating and fruitful area of research. The

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famous Ryser–Brualdi–Stein conjecture asserts that every $n \times n$ Latin square has a partial transversal of size $n-1$, which is equivalent to saying that any proper n-edge-colouring of $K_{n,n}$ admits a rainbow matching of size $n-1$.

As a second example, we consider a powerful application of rainbow colourings to graph decompositions. Graph decomposition problems are central problems in graph theory with a long history, and many fundamental questions are still unanswered. We say that $H_1, \ldots, H_t$ *decompose* G if $H_1, \ldots, H_t$ are edge-disjoint subgraphs of G covering every edge of G. Perhaps one of the oldest decomposition results is Walecki's theorem from 1892 saying that K_{2n+1} can be decomposed into Hamilton cycles. His construction not only gives any decomposition, but a 'cyclic' decomposition based on a rotation technique, by finding one Hamilton cycle H^* in K_{2n+1} and a permutation π on $V(K_{2n+1})$ such that the permuted copies $\pi^i(H^*)$ of H^* for $i = 0, \ldots, n-1$ are pairwise edge-disjoint (and thus decompose K_{2n+1}). The difficulty here is of course finding H^* given π, or vice versa. Unfortunately, for many other decomposition problems, this is not as easy, or indeed not possible at all. In recent years, some exciting progress has been made in the area of (hyper-)graph decompositions, for example Keevash's proof of the Existence conjecture [**16**] and generalizations thereof [**12, 13, 17**], progress on the Gyárfás–Lehel tree-packing conjecture [**14**] and the resolution of the Oberwolfach problem [**11**]. Those results are based on very different techniques, such as absorbing-type methods, randomised constructions and variations of Szemeredi's regularity technique. In a recent paper, Montgomery, Pokrovskiy and Sudakov [**28**] brought the use of the rotation technique back into focus when proving an old conjecture of Ringel approximately, by reducing it to a rainbow embedding problem. A similar approach has previously been used by Drmota and Lladó [**8**] in connection with a bipartite version of Ringel's conjecture posed by Graham and Häggkvist. Ringel conjectured in 1963 that any tree with n edges decomposes K_{2n+1}. A strengthening of Ringel's conjecture is due to Kotzig [**24**], who conjectured in 1973 that there even exists a cyclic decomposition. This can be phrased as a rainbow embedding problem as follows: Order the vertices of K_{2n+1} cyclically and colour each edge $\{i, j\} \in E(K_{2n+1})$ with its distance (that is, the distance of i, j in the cyclic ordering), which is a number between 1 and n. The simple but crucial observation is that if T is a rainbow subtree, then T can be rotated according to the cyclic vertex ordering, yielding $2n+1$ edge-disjoint copies of T (and thus a cyclic decomposition if T has n edges). Note that for each vertex v and any given distance, there are only two vertices which have exactly this distance from v. More generally, an edge-colouring is called *locally k-bounded* if each colour class has maximum degree at most k. The following statement thus implies Kotzig's and Ringel's conjecture: Any locally 2-bounded edge-colouring of K_{2n+1} contains a rainbow copy of any tree with n edges. Montgomery, Pokrovskiy and Sudakov [**28**] proved the following asymptotic version of this statement (all asymptotic terms are considered as $n \to \infty$), which in turn yields an asymptotic version of the conjectures.

Theorem 1.1 ([**28**]). *For fixed k, any locally k-bounded edge-colouring of K_n contains a rainbow copy of any tree with $(1 - o(1))n/k$ edges.*

Our main result is very similar in spirit. Roughly speaking, instead of dealing with trees, our results apply to general graphs H, but we require H to have bounded degree, whereas one of the great achievements of [**28**] is that no such requirement is necessary when dealing with trees. As a consequence, our main result implies asymptotic versions of analogue conjectures to Ringel's and Kotzig's conjecture where trees with n edges are replaced by general bounded degree graphs with n edges. The following is a special case of our main result (Theorem 2.1). An edge-colouring is called *(globally) k-bounded* if any colour appears at most k times.

Theorem 1.2. *Suppose H is a graph on at most n vertices with $\Delta(H) = \mathcal{O}(1)$. Then any locally $\mathcal{O}(1)$-bounded and globally $(1 - o(1))\binom{n}{2}/e(H)$-bounded edge-colouring of K_n contains a rainbow copy of H.*

It is plain that any locally k-bounded colouring is (globally) $kn/2$-bounded. Thus, Theorem 1.2 implies Theorem 1.1 for bounded-degree trees. Note that the assumption that the colouring is $(1 - o(1))\binom{n}{2}/e(H)$-bounded is asymptotically best possible in the sense that if the colouring was not $\binom{n}{2}/e(H)$-bounded, there might be less than $e(H)$ colours, making the existence of a rainbow copy of H impossible.

Rainbow embedding problems have also been extensively studied for their own sake. For instance, Erdős and Stein asked for the maximal k such that any k-bounded edge-colouring of K_n contains a rainbow Hamilton cycle (cf. [**9**]). After several subsequent improvements, Albert, Frieze and Reed [**1**] showed that $k = \Omega(n)$. Theorem 1.2 implies that under the additional assumption that the colouring is locally $\mathcal{O}(1)$-bounded, we have $k = (1 - o(1))n/2$, which is essentially best possible. This is not a new result but also follows from results in [**18, 27**]. However, the results in [**18, 27**] are limited to finding Hamilton cycles or F-factors (in fact, approximate decompositions into these structures). Theorem 1.2 allows the same conclusion if we seek an $\sqrt{n/2}\times\sqrt{n/2}$ grid, say, or any other bounded-degree graph with roughly n edges. For general subgraphs H, the best previous result is due to Böttcher, Kohayakawa, and Procacci [**3**], who showed that given any $n/(51\Delta^2)$-bounded edge-colouring of K_n and any graph H on n vertices with $\Delta(H) \leq \Delta$, one can find a rainbow copy of H. Our Theorem 1.2 improves, for bounded-degree graphs, the global boundedness condition to an asymptotically best possible one, under the additional assumption that the colouring is $\mathcal{O}(1)$-bounded.

2. Main result

We now state a more general version of Theorem 1.2. We say that a graph G on n vertices is *(ε, d)-quasirandom* if for all $v \in V(G)$ we have $\deg_G(v) = (d \pm \varepsilon)n$, and for all disjoint $S, T \subseteq V(G)$ with $|S|, |T| \geq \varepsilon n$, we have $e_G(S, T) = (d \pm \varepsilon)|S||T|$.

Theorem 2.1. *For all $d, \gamma, \Delta, \Lambda$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose G and H are graphs on n vertices, G is (ε, d)-quasirandom and $\Delta(H) \leq \Delta$. Then given any locally Λ-bounded and globally $(1 - \gamma)e(G)/e(H)$-bounded edge-colouring of G, the graph G contains a rainbow copy of H.*

Clearly, Theorem 2.1 implies Theorem 1.2. We derive Theorem 2.1 from an even more general 'blow-up lemma' (Lemma 2.2). The original blow-up lemma of Komlós, Sárközy and Szemerédi [**22**] developed roughly 20 years ago, is a powerful tool to find spanning subgraphs and has found numerous important applications in extremal combinatorics [**5, 11, 20, 21, 23, 25, 26**]. Roughly speaking, it says that given a k-partite graph G that is 'super-regular' between any two vertex classes, and a k-partite bounded-degree graph H with a matching vertex partition, then H is a subgraph of G. Note that the conclusion is trivial if G is complete k-partite, so the crux here is that instead of requiring G to be complete between any two vertex classes, super-regularity suffices. Such a scenario can often be obtained in conjunction with Szemerédi's regularity lemma, which makes it widely applicable. Many variations of the blow-up lemma have been obtained over the years (e.g. [**2, 4, 7, 15, 19, 29**]). Recently, the second and third author [**10**] proved a rainbow blow-up lemma for $o(n)$-bounded edge-colourings which allows to find a rainbow embedding of H. The present paper builds upon this result. The key novelty is that instead of requiring the colouring to be $o(n)$-bounded, our new result applies for almost optimally bounded colourings. (But we assume here that the colouring is locally $\mathcal{O}(1)$-bounded, which is not necessary in [**10**]).

In order to state our new rainbow blow-up lemma, we need to introduce some terminology. If $c\colon E(G) \to C$ is an edge-colouring of a graph G and $\alpha \in C$, denote by $e^{\alpha}(G)$ the number of α-coloured edges of G, and denote by $e^{\alpha}_G(S,T)$ the number of α-coloured edges of G with one endpoint in S and the other one in T. Define $d_G(S,T) := e_G(S,T)/|S||T|$ as the *density* of the pair S,T in G. We say that the bipartite graph $G[V_1,V_2]$ is (ε,d)-*super-regular* if

- for all $S \subseteq V_1$ and $T \subseteq V_2$ with $|S| \geq \varepsilon|V_1|$, $|T| \geq \varepsilon|V_2|$, we have $|d_G(S,T) - d| \leq \varepsilon$;
- for all $i \in [2]$ and $v \in V_i$, we have $|N_G(v) \cap V_{3-i}| = (d \pm \varepsilon)|V_{3-i}|$.

We say that $(H, G, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ is an (ε,d)-*super-regular blow-up instance* if

- H and G are graphs, $(X_i)_{i\in[r]}$ is a partition of $V(H)$ into independent sets, $(V_i)_{i\in[r]}$ is a partition of $V(G)$, and $|X_i| = |V_i|$ for all $i \in [r]$, and
- for all $ij \in \binom{[r]}{2}$, the bipartite graph $G[V_i,V_j]$ is (ε,d)-super-regular.

We say that $\phi\colon V(H) \to V(G)$ is an *embedding of H into G* if ϕ is injective and $\phi(x)\phi(y) \in E(G)$ for all $xy \in E(H)$. We also write $\phi\colon H \to G$ in this case. We say that ϕ is *rainbow* if $\phi(H)$ is rainbow.

We now state our new rainbow blow-up lemma.

Lemma 2.2 (Rainbow blow-up lemma). *For all $d, \gamma, \Delta, \Lambda, r$, there exists an $\varepsilon > 0$ and an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose $(H, G, (X_i)_{i\in[r]}, (V_i)_{i\in[r]})$ is an (ε,d)-super-regular blow-up instance. Assume further that*

(i) $\Delta(H) \leq \Delta$;
(ii) $|V_i| = (1 \pm \varepsilon)n$ *for all* $i \in [r]$;

(iii) $c\colon E(G) \to C$ *is a locally* Λ*-bounded edge-colouring such that the following holds for all* $\alpha \in C$*:*

$$\sum_{ij \in \binom{[r]}{2}} e^{\alpha}_{G}(V_i, V_j) e_H(X_i, X_j) \leq (1-\gamma) dn^2.$$

Then there exists a rainbow embedding ϕ *of* H *into* G *such that* $\phi(x) \in V_i$ *for all* $i \in [r]$ *and* $x \in X_i$*.*

The boundedness condition in (iii) can often be simplified, for instance in the following natural situations: In the proof of Theorem 2.1, we randomly partition $V(G)$ into equal-sized $(V_i)_{i\in[r]}$. For each colour $\alpha \in C$, we then have that $e^{\alpha}_{G}(V_i, V_j)$ is roughly the same for all $ij \in \binom{[r]}{2}$, and so c needs to be $(1-\gamma)e(G[V_1, \ldots, V_r])/e(H)$-bounded.

Similarly, if c is 'colour-split', that is, $e^{\alpha}_{G}(V_i, V_j) \in \{e^{\alpha}(G), 0\}$, then c needs to be $(1-\gamma)e(G[V_i, V_j])/e(H[X_i, X_j])$-bounded for all $ij \in \binom{[r]}{2}$. Both conditions are easily seen to be asymptotically best possible.

The blow-up lemma for $o(n)$-bounded colourings was applied in [**10**] to transfer the bandwidth theorem to the rainbow setting, using Szemerédi's regularity lemma. Unfortunately, it seems much more complicated to use the regularity lemma in the optimally bounded setting.

3. Applications

Rainbow edge-colourings can be utilized to find decompositions of graphs into smaller graphs. The general idea is based on a group action on the host graph and has frequently been used in the recent literature; for example by Montgomery, Pokrovskiy and Sudakov for an approximate solution of Ringel's conjecture [**28**], by Drmota and Lladó regarding a conjecture of Graham and Häggkvist [**8**], and by Bryant and Scharaschkin solving the Oberwolfach problem for infinitely many orders [**6**]. We give two examples where our main result is applied to obtain approximate decompositions into bounded degree graphs of the complete (bipartite) graph.

Theorem 3.1. *For all* $\Delta \in \mathbb{N}$*, there exist* $\varepsilon > 0$ *and* $n_0 \in \mathbb{N}$ *such that the following holds for all* $n \geq n_0$*. Suppose* H *is a graph with* $(1-\varepsilon)n/2$ *edges with* $\Delta(H) \leq \Delta$ *and* $|V(H)| \leq n$*. Then* K_n *contains* n *edge-disjoint copies of* H*.*

Theorem 3.2. *For all* $\Delta \in \mathbb{N}$*, there exist* $\varepsilon > 0$ *and* $n_0 \in \mathbb{N}$ *such that the following holds for all* $n \geq n_0$*. Suppose* H *is a graph with* $\Delta(H) \leq \Delta$ *and at most* $(1-\varepsilon)n$ *edges, and* $V(H)$ *is partitioned into* 2 *independent sets of size* n*. Then the complete bipartite graph* $K_{n,n}$ *contains* n *edge-disjoint copies of* H*.*

Approximate decomposition results which do not arise from a group action but from random procedures have been studied recently in great depth. To the expense that one does not obtain very symmetric (approximate) decompositions, it is possible to embed different graphs and not only many copies of a single graph. In particular, the blow-up lemma for approximate decompositions by Kim, Kühn, Osthus and Tyomkyn [**19**] yields approximate decompositions into bounded degree graphs of quasirandom multipartite graphs very similarly to our setting, which in particular implies Theorem 3.1 and Theorem 3.2.

Beyond the approximate solution of Ringel's conjecture, Montgomery, Pokrovskiy and Sudakov also provide applications of their result to graph labelling and orthogonal double covers. Our main result yields similar applications for bounded degree graphs which can be essentially proved analogously. We refer to the full-length version of this manuscript for a detailed discussion.

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