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COLORING TRIANGLE-FREE L-GRAPHS WITH $O(\log\log n)$ COLORS

B. WALCZAK

Abstract. It is proved that triangle-free intersection graphs of n L-shapes in the plane have chromatic number $O(\log\log n)$. This improves the previous bound of $O(\log n)$ (McGuinness, 1996) and matches the known lower bound construction (Pawlik et al., 2013).

1. Introduction

The *intersection graph* of a family of sets $\mathcal{F}$ has these sets as vertices and the pairs of the sets that intersect as edges. An *L-shape* is a set in the plane formed by one horizontal segment and one vertical segment joined at the left endpoint of the former and the bottom endpoint of the latter, as in the letter L. An *L-graph* is an intersection graph of L-shapes. A stretching argument of Middendorf and Pfeiffer [**14**] shows that L-graphs form a subclass of the *segment graphs*, that is, intersection graphs of straight-line segments in the plane. Segment graphs form a subclass of the *string graphs* – intersection graphs of generic curves in the plane.

L-graphs are perhaps not as natural as segment graphs, but they capture a lot of complexity of segment graphs while being significantly easier to deal with. For instance, the famous result of Chalopin and Gonçalves [**1**] that all planar graphs are segment graphs (solution to Scheinerman's conjecture) was recently strengthened by Gonçalves, Isenmann and Pennarun [**4**] who showed, with a much simpler and more elegant argument, that all planar graphs are L-graphs. Other recent works on L-graphs and their relation to other classes of graphs include [**2, 7**].

Our concern in this paper is how large the chromatic number χ can be in terms of the number of vertices n for triangle-free geometric intersection graphs. Classical constructions of triangle-free graphs with arbitrarily large chromatic number such as Mycielski graphs and shift graphs achieve $\chi = \Theta(\log n)$ but are not realizable as string graphs. Non-constructive methods even provide triangle-free graphs with $\chi = \Theta(\sqrt{n/\log n})$ [**8**]. However, for triangle-free geometric intersection graphs (string graphs), even to determine whether the chromatic number can grow arbitrarily high was a long-standing open problem. Raised in the 1980s by Erdős

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for segment graphs (see [**5**]) and by Gyárfás and Lehel [**6**] for L-graphs, it was solved only recently.

Theorem 1 (Pawlik et al. [**15**])**.** *There exist triangle-free L-graphs with chromatic number $\Theta(\log\log n)$.*

By contrast, McGuinness [**11**] proved that *infinite-L-graphs* (intersection graphs of L-shapes whose vertical parts are upward-infinite) have chromatic number bounded in terms of the clique number.

The key insights that led to the construction in Theorem 1 came from analyzing *directed frame graphs*, that is, intersection graphs of *frames* (boundaries of axis-parallel rectangles) whose top sides are free of intersections with other frames. They form a subclass of the L-graphs, and the construction in Theorem 1 actually provides triangle-free directed frame graphs with $\chi = \Theta(\log\log n)$. The same key insights led to the proof of the following upper bound.

Theorem 2 (Krawczyk, Pawlik and Walczak [**9**])**.** *Triangle-free directed frame graphs have chromatic number $O(\log\log n)$.*

The same bound also holds for general triangle-free frame graphs (not necessarily directed) [**9**].

Our contribution is a generalization of Theorem 2 to triangle-free L-graphs.

Theorem 3. *Triangle-free L-graphs have chromatic number $O(\log\log n)$.*

The previous best bound was $\chi = O(\log n)$; it follows directly from the above-mentioned result of McGuinness [**11**] on infinite-L-graphs and is valid also for L-graphs with clique number bounded by any constant. When the clique number is bounded by ω, the best known upper bounds on the chromatic number are $O((\log\log n)^{\omega-1})$ for frame graphs [**10**], $O(\log n)$ for segment graphs [**17**], and $(\log n)^{O(\log\omega)}$ for string graphs in general [**3**]. Furthermore, there exist string graphs (not realizable as segment graphs) with clique number ω and chromatic number $\Theta((\log\log n)^{\omega-1})$ [**10**]. It remains open whether a double-logarithmic upper bound on the chromatic number is valid, for instance, for triangle-free segment graphs or for L-graphs with bounded clique number.

2. Terminology and preliminary results

Let $h(\ell)$ and $v(\ell)$ denote the horizontal and the vertical segment of an L-shape ℓ, respectively. Their intersection point is the *corner* of ℓ, and their other endpoints are the *right* and the *top endpoint* of ℓ, respectively. We will be assuming that the horizontal segments of all L-shapes that we consider have distinct y-coordinates and the vertical ones have distinct x-coordinates.

A point p (plane set r) lies *above*/*below*/*to the left*/*right of* a plane set s if the ray emanating from p (every point of r) downwards/upwards/rightwards/leftwards intersects s. A transformation of the plane called *horizontal*/*vertical shifting* with respect to a horizontal/vertical segment s, illustrated in Figure 1, preserves essential combinatorial structure of a family of L-shapes while avoiding any corners

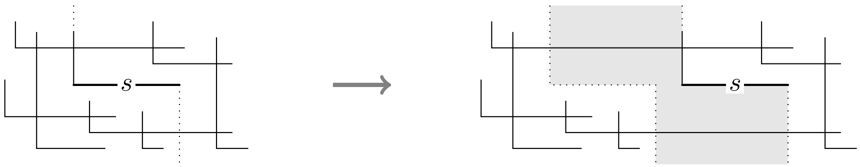

Figure 1. Horizontal shifting with respect to a horizontal segment s.

or endpoints below/to the left of s. It can be performed when none of the L-shapes crosses s.

Graph-theoretic terms like *chromatic number* and *triangle-free* applied directly to a family of curves $\mathcal{F}$ have the same meaning as when applied to the intersection graph of $\mathcal{F}$. A family of curves $\mathcal{F}$ is 1-*intersecting* if any two curves in $\mathcal{F}$ have at most one common point. For $c \in \mathcal{F}$, let $\mathcal{F}(c)$ denote the family of curves at distance exactly 2 from c in the intersection graph of $\mathcal{F}$.

Lemma 4 (McGuinness [**13**, Theorem 5.3])**.** *There is $\alpha > 0$ such that every triangle-free 1-intersecting family of curves $\mathcal{F}$ satisfies $\chi(\mathcal{F}) \leq \alpha \max_{c\in\mathcal{F}} \chi(\mathcal{F}(c))$.*

Let ℓ_0 be a horizontal line. A curve c is *grounded to* ℓ_0 when one endpoint of c lies on ℓ_0 and the remaining part of c lies above ℓ_0.

Lemma 5 (McGuinness [**12**])**.** *Triangle-free 1-intersecting families of curves grounded to a fixed horizontal line ℓ_0 have bounded chromatic number.*

An ℓ_0-*even-curve* is a curve that starts above ℓ_0 and crosses ℓ_0 properly a positive even number of times, ending again above ℓ_0. The two parts of an ℓ_0-even-curve c from an endpoint to the first intersection point with ℓ_0 are denoted by $L(c)$ and $R(c)$ so that the common point of $L(c)$ with ℓ_0 is to the left of that of $R(c)$. A family of ℓ_0-even-curves is an *LR-family* if every intersection between two of its members c_1 and c_2 is between $L(c_1)$ and $R(c_2)$ or vice versa.

Lemma 6 (Rok and Walczak [**16**, Theorem 4])**.** *Triangle-free LR-families of ℓ_0-even-curves have bounded chromatic number.*

3. Proof of Theorem 3

Let $\mathcal{F}$ be a triangle-free family of n L-shapes. Lemma 4 yields $\chi(\mathcal{F}) = O(\chi(\mathcal{F}(\ell_0)))$ for some $\ell_0 \in \mathcal{F}$, so it suffices to prove $\chi(\mathcal{F}(\ell_0)) = O(\log\log n)$. Every L-shape $\ell \in \mathcal{F}(\ell_0)$ has a *support* – an L-shape in $\mathcal{F}$ that crosses ℓ_0 and ℓ. The supports are pairwise disjoint, as they cross ℓ_0 and $\mathcal{F}$ is triangle-free. For $k \in \{1, \dots, 6\}$, let $\mathcal{F}^k(\ell_0)$ be the L-shapes in $\mathcal{F}(\ell_0)$ such that

$k = 1$: some support crosses $h(\ell_0)$ and $h(\ell)$, and $h(\ell)$ lies above the horizontal line containing $h(\ell_0)$,

$k = 2$: some support crosses $h(\ell_0)$ and $h(\ell)$, and $h(\ell)$ lies below the horizontal line containing $h(\ell_0)$,

$k = 3$: some support crosses $h(\ell_0)$ and $v(\ell)$ but no support crosses $h(\ell)$,

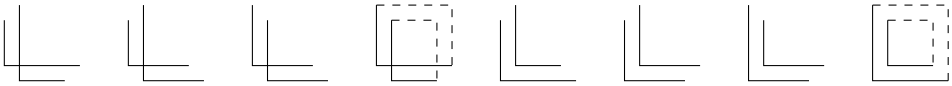

Figure 2. Configurations of pairs L-shapes with intersecting bounding boxes; the case of right endpoints with common x-coordinate is included in (b), (c), (e), (h), and the case of top endpoints with common y-coordinate in (c), (d), (g), (h).

$k = 4$: some support crosses $v(\ell_0)$ and $v(\ell)$, and $v(\ell)$ lies to the right of the vertical line containing $v(\ell_0)$,
$k = 5$: some support crosses $v(\ell_0)$ and $v(\ell)$, and $v(\ell)$ lies to the left of the vertical line containing $v(\ell_0)$,
$k = 6$: some support crosses $v(\ell_0)$ and $h(\ell)$ but no support crosses $v(\ell)$.

It follows that $\mathcal{F}(\ell_0) = \mathcal{F}^1(\ell_0) \cup \cdots \cup \mathcal{F}^6(\ell_0)$. We will prove that $\chi(\mathcal{F}^k(\ell_0)) = O(\log\log n)$ for $k \in \{1, 3, 4, 6\}$, and then we will deal with the cases $k \in \{2, 5\}$ by a recursive argument.

First, we deal with the case $k = 1$. Let $\mathcal{L} = \mathcal{F}^1(\ell_0)$. Let $\mathcal{S}$ be the L-shapes in $\mathcal{F}$ that cross $h(\ell_0)$. Since the L-shapes in $\mathcal{L}$ are disjoint from ℓ_0 and lie entirely above the horizontal line containing $h(\ell_0)$, we can simplify the setting as follows: we replace ℓ_0 by that horizontal line (called ℓ_0 henceforth), and we replace all L-shapes in $\mathcal{S}$ by their parts that lie above ℓ_0 (called *supports* henceforth). The family $\mathcal{L} \cup \mathcal{S}$ of L-shapes and vertical segments is triangle-free.

Figure 2 illustrates possible configurations of pairs of L-shapes whose minimal bounding boxes intersect. If $\mathcal{L}$ contains no pair of L-shapes in configurations (a)–(c) and (e)–(g), then completing the L-shapes in $\mathcal{L}$ to frames (by adding the right and top sides and taking proper care of collinear sides) shows that the intersection graph of $\mathcal{L}$ is a directed frame graph, and we can apply Theorem 2 to conclude that $\chi(\mathcal{L}) = O(\log\log n)$. Therefore, our goal will be to reduce the problem to the case where configurations (a)–(c) and (e)–(g) are excluded. The reduction will keep modifying $\mathcal{L}$ and $\mathcal{S}$ to make them satisfy more and more additional conditions. We will present each step of the reduction by first formulating a new condition that $\mathcal{L}$ and $\mathcal{S}$ should satisfy and then explaining how to modify $\mathcal{L}$ and $\mathcal{S}$ to ensure that condition while preserving all previous conditions and *changing* $\chi(\mathcal{L})$ *by at most a constant factor*. The latter guarantees that the bound $\chi(\mathcal{L}) = O(\log\log n)$ after the reduction implies the same bound for the original family $\mathcal{L}$.

The *handle* of an L-shape $\ell \in \mathcal{L}$ is the part of ℓ between the top endpoint of ℓ and the leftmost intersection point of ℓ with a support, and the *hook* of ℓ is the part of ℓ to the right of the rightmost intersection point of ℓ with a support.

Condition 1. No two handles of L-shapes in $\mathcal{L}$ intersect.

The family $\mathcal{H}$ of handles of the L-shapes in $\mathcal{L}$ can be transformed into a 1-intersecting family of curves grounded to ℓ_0 with the same intersection graph by connecting them to ℓ_0 along the leftmost supports, as illustrated in Figure 3 (a). Therefore, by Lemma 5, $\chi(\mathcal{H})$ is bounded. Let $\mathcal{L}_c$ denote the L-shapes in $\mathcal{L}$ whose

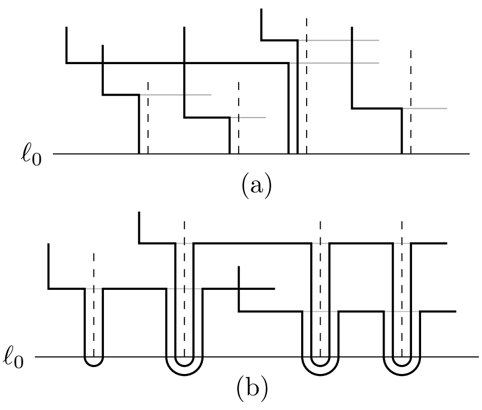

Figure 3. Transformations of L-shapes to curves considered in the proof.

handles have color c in an optimal proper coloring of $\mathcal{H}$. Then $\chi(\mathcal{L}) \leq \sum_c \chi(\mathcal{L}_c) = O(\chi(\mathcal{L}_{c^\star}))$, where $\chi(\mathcal{L}_{c^\star}) = \max_c \chi(\mathcal{L}_c)$. We set $\mathcal{L} := \mathcal{L}_{c^\star}$, and Condition 1 holds.

Condition 2. All of the L-shapes in $\mathcal{L}$ have empty hooks. Consequently, no two L-shapes in $\mathcal{L}$ occur in configuration (b) or (c) from Figure 2.

Let $\mathcal{L}'$ be the family of L-shapes obtained from $\mathcal{L}$ by cutting all hooks off. Consider a color class $\mathcal{L}'_c$ in an optimal proper coloring of $\mathcal{L}'$. Add all hooks back to the members of $\mathcal{L}'_c$ to obtain a family $\mathcal{L}_c \subseteq \mathcal{L}$ with handle-hook intersections only. The family $\mathcal{L}_c$ can be transformed into an LR-family of ℓ_0-even-curves with the same intersection graph, as illustrated in Figure 3 (b). Therefore, by Lemma 6, $\chi(\mathcal{L}_c)$ is bounded. This yields $\chi(\mathcal{L}) \leq \sum_c \chi(\mathcal{L}_c) = O(\chi(\mathcal{L}'))$. If two L-shapes $\ell_1, \ell_2 \in \mathcal{L}'$ occurred in configuration (b) or (c) from Figure 2, they would form a triangle with the rightmost support of ℓ_1 or ℓ_2. Therefore, Condition 2 holds after we set $\mathcal{L} := \mathcal{L}'$.

Condition 3. No L-shape in $\mathcal{L}$ has corner or endpoint below the handle of any other L-shape in $\mathcal{L}$.

By Condition 1, no member of $\mathcal{L} \cup \mathcal{S}$ crosses the horizontal segment of the handle of any L-shape in $\mathcal{L}$, so Condition 3 can be ensured by horizontal shifting with respect to each of these segments.

Condition 4. If $\ell_1 \in \mathcal{L} \cup \mathcal{S}$ and $\ell_2 \in \mathcal{L}$ intersect, then $v(\ell_2)$ does not lie entirely to the left of ℓ_1. In particular, no two members of $\mathcal{L}$ occur in configuration (a) from Figure 2.

Let $\mathcal{L}'$ be the family of those $\ell \in \mathcal{L}$ for which there is a *witness* $\ell_1 \in \mathcal{L} \cup \mathcal{S}$ such that $v(\ell)$ lies to the left of ℓ_1. No two members of $\mathcal{L}'$ intersect; otherwise they would form configuration (a) or (d) from Figure 2, so they would form a triangle with the witness of the one of them with lower horizontal segment. Thus $\chi(\mathcal{L}) \leq \chi(\mathcal{L} \smallsetminus \mathcal{L}') + 1$, and Condition 4 holds after setting $\mathcal{L} := \mathcal{L} \smallsetminus \mathcal{L}'$.

Condition 5. No two L-shapes in $\mathcal{L}$ form configuration (e) or (f) from Figure 2.

Suppose $\ell_1, \ell_2 \in \mathcal{L}$ occur in configuration (e) or (f), where ℓ_1 has lower horizontal segment. Modify ℓ_1 by pulling its top endpoint up onto the horizontal line containing the top endpoint of ℓ_2. This can create additional intersections between ℓ_1 and other members of $\mathcal{L}$, but it is not difficult to see that $\mathcal{L} \cup \mathcal{S}$ remains triangle-free and Conditions 1–4 are preserved (we omit the details). Repeating this modification for all "bad" pairs must terminate (because the set of horizontal lines containing the top endpoints of the L-shapes in $\mathcal{L}$ does not change), ensuring Condition 5.

Condition 6. No corner of an L-shape in $\mathcal{L}$ lies to the left of two intersecting members of $\mathcal{L} \cup \mathcal{S}$.

Suppose that $\ell_1 \in \mathcal{L} \cup \mathcal{S}$ crosses $h(\ell_2)$ for some $\ell_2 \in \mathcal{L}$. Let s be the part of $v(\ell_2)$ that lies to the left of ℓ_1. No L-shape in $\mathcal{L}$ can cross s, otherwise (by Condition 2) it would also cross ℓ_1, thus creating a triangle. Vertical shifting with respect to s ensures that no L-shape in $\mathcal{L}$ has its corner to the left of s while preserving Conditions 1–5.

Condition 7. No two L-shapes in $\mathcal{L}$ occur in configuration (g) from Figure 2.

Suppose $\ell_1, \ell_2 \in \mathcal{L}$ occur in configuration (g), where ℓ_1 has lower horizontal segment. Modify ℓ_1 by pulling its right endpoint further to the right onto the rightmost support of ℓ_2. This can create additional intersections between ℓ_1 and other members of $\mathcal{L} \cup \mathcal{S}$, but it is not difficult to see that $\mathcal{L} \cup \mathcal{S}$ remains triangle-free and Conditions 1–6 are preserved (we omit the details). Repeating this modification for all "bad" pairs must terminate, ensuring Condition 7.

Now, of all configurations illustrated in Figure 2, only (d) and (h) can occur in $\mathcal{L}$. Therefore, we can complete the L-shapes in $\mathcal{L}$ to frames without changing the intersection graph, and we derive the bound $\chi(\mathcal{L}) = O(\log \log n)$ from Theorem 2. This completes the proof for the case $k = 1$.

Now, we deal with the case $k = 6$. Let $\mathcal{L} = \mathcal{F}^6(\ell_0)$. Let $\mathcal{S}$ be the supports of the L-shapes in $\mathcal{L}$. For each $s \in S$, since no L-shape in $\mathcal{L}$ crosses $h(s)$, we can perform horizontal shifting with respect to $h(s)$. Then, we replace each $s \in \mathcal{S}$ by the vertical extension of $v(s)$ down to some common horizontal line, and we replace ℓ_0 by that line, ending up in the same setting as in the case $k = 1$.

By symmetry, the case $k = 4$ is analogous to $k = 1$, and the case $k = 3$ is analogous to $k = 6$.

It remains to deal with the cases $k \in \{2, 5\}$. Let $\mathcal{F}_0 = \mathcal{F}$. Repeating the arguments above for $i = 0, 1, 2$, we find $\ell_i \in \mathcal{F}_i$ and $k_i \in \{2, 5\}$ so that $\chi(\mathcal{F}_i) = O(\chi(\mathcal{F}_i(\ell_i))) = O(\log \log n + \chi(\mathcal{F}_{i+1}))$ where $\mathcal{F}_{i+1} = \mathcal{F}_i^{k_i}(\ell_i)$. Let $i, j \in \{0, 1, 2\}$ be such that $i < j$ and $k_i = k_j$. Suppose $k_i = k_j = 2$. Let $\mathcal{S}$ be the L-shapes in $\mathcal{F}_j$ that cross $h(\ell_j)$. Every $\ell \in \mathcal{F}_3 \subseteq \mathcal{F}_{j+1}$ has a support in $\mathcal{S}$ that crosses $h(\ell)$, and using these supports, we can define the *handle* and the *hook* of ℓ like we did when considering the case $k = 1$. Since $\mathcal{S} \subseteq \mathcal{F}_i^2(\ell_i)$, every $s \in \mathcal{S}$ has a support that crosses $h(\ell_i)$ and $h(s)$ but not ℓ_j (otherwise it would form a triangle with s and ℓ_j). Thus every $s \in \mathcal{S}$ crosses the vertical ray emanating from the right endpoint of ℓ_j downwards. This easily implies that every crossing between two L-shapes in $\mathcal{F}_3$ is a handle-hook crossing (we omit the details). Therefore, we can apply Lemma 6 like we did when considering Condition 2 of the case $k = 1$ to conclude that $\chi(\mathcal{F}_3)$ is bounded and thus $\chi(\mathcal{F}) = O(\log\log n + \chi(\mathcal{F}_3)) = O(\log\log n)$. This completes the proof for the case $k_i = k_j = 2$. The case $k_i = k_j = 5$ is analogous, by symmetry.

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B. Walczak, Department of Theoretical Computer Science, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland,

e-mail: `walczak@tcs.uj.edu.pl`