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SPREADING LINEAR TRIPLE SYSTEMS AND EXPANDER TRIPLE SYSTEMS

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ABSTRACT. The existence of Steiner triple systems STS(n) of order n containing no nontrivial subsystem is well known for every admissible n. We generalize this result in two ways. First we define the expander property of 3-uniform hypergraphs and show the existence of Steiner triple systems which are almost perfect expanders.

Next we define the strong and weak spreading property of linear hypergraphs, and determine the minimum size of a linear triple system with these properties, up to a small constant factor. A linear triple system on a vertex set V has the spreading, or respectively weakly spreading property if any sufficiently large subset $V' \subset V$ contains a pair of vertices with which a vertex of $V \setminus V'$ forms a triple of the system. Here the condition on V' refers to $|V'| \geq 4$ or V' is the support of more than one triples, respectively. This property is strongly connected to the connectivity of the structure the so-called influence maximisation. We also discuss how the results are related to Erdős' conjecture on locally sparse STSs, subsquare-free Latin-squares and possible applications in finite geometry.

1. Introduction

A Steiner triple system $\mathcal{S}$ of order n (briefly STS(n)) consists of an n-element set V and a collection of triples (or blocks) of V, such that every pair of distinct points in V is contained in a unique block. It is well known that there exists a STS(n) if and only if $n \equiv 1, 3 \pmod 6$ [**12**], these values are called *admissible*. Steiner triple systems correspond to triangle decompositions of the complete graph $G = K_n$. In the context of triangle decompositions of a graph G, an edge will always refer to a pair of vertices which is contained in one triple of a certain triple system, $E(G)$ denotes the edge set of G, while $|\mathcal{S}|$ is the number of triples in the system, which obviously equals $\frac{1}{3}|E(G)|$ in the case of triple systems obtained from triangle decompositions of a graph G.

A nontrivial Steiner subsystem of $\mathcal{S}$ is a STS(n') induced by a proper subset of V, with $n' > 3$. Speaking about a subsystem, we always suppose that it is of order greater than 3. Similarly, we call a subset $V' \subset V$ of the underlying set of a

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triple system $\mathcal{F}$ *nontrivial* if it has size at least 3 and it is not a triple of the triple system.

This paper is devoted to the study of two main features of certain linear triples systems, also called linear 3-graphs. The first property is the *expander property* while the second is the so-called *spreading property.*

In 1973, Erdős formulated the following conjecture.

Conjecture 1.1 (Erdős, [**8**])**.** For every $k \geq 2$ there exists a threshold n_k such that for all admissible $n > n_k$, there exists a Steiner triple system of order n with the following property: every $t+2$ vertices induce less than t triples of $\mathcal{S}$ for $2 \leq t \leq k$.

This conjecture is still open, although recently Glock, Kühn, Lo and Osthus [**11**] and independently Bohman and Warnke [**3**] proved its asymptotic version. In other words, this conjecture asserts the existence of arbitrarily sparse Steiner triple systems.
One should note here that it is also a question whether typical Steiner triple systems are sparse in a very robust sense, namely that they do not contain Steiner subsystems. Indeed, this is equivalent to avoid a set of $t < n$ vertices inducing quadratically many, $\frac{1}{3}\binom{t}{2}$ triples. The first result in this direction was due to Doyen [**6**], who proved the existence of *at least one* subsystem-free STS(n) for every admissible order n. In the language of decompositions, a subsystem-free STS may be seen as a triangle decomposition of the edge set where every subset $V' \subset V(G)$ contains at least one edge which belongs to a triangle not induced by V'. In order to capture this phenomenon and its generalisation, we require some notation and definitions.

Definition 1.2. Given a 3-uniform linear hypergraph $\mathcal{F}$ (i.e. linear triple system), let $E(\mathcal{F})$ be the collection of vertex pairs (x, y) for which there exists a triple (x, y, z) from the system $\mathcal{F}$, containing x and y. The corresponding graph $G(\mathcal{F})$ is referred to as the *shadow of the system.*

Hence the shadow is simply the underlying graph of the triple system.

Definition 1.3. Consider a graph $G = G(V, E)$ that admits a triangle decomposition. This decomposition corresponds to a linear triple system $\mathcal{F}$. For an arbitrary set $V' \subset V$, $N(V')$ denotes the set of its *neighbours*:

$$z \in N(V') \Leftrightarrow z \in V \setminus V' \text{ and } \exists xy \in E(G[V']) : \{x, y, z\} \in \mathcal{F}.$$

The *closure* $\mathrm{cl}(V')$ of a subset V' w.r.t. a (linear) triple system $\mathcal{F}$ is the smallest set $W \supseteq V'$ for which $|N(W)| = 0$ holds. Note that the closure uniquely exists for each set V'. We call a (linear) triple system $\mathcal{F}$ *spreading* if $\mathrm{cl}(V') = V$ for every nontrivial subset $V' \subset V$.

Consequently, a STS(n) is subsystem-free if and only if $|N(V')| > 0$ holds for all nontrivial subsets V' of the underlying set V of the system. Doyen used the term non-degenerate plane for STSs with the spreading property [**6, 7**].

Two natural questions arise here. The first one concerns the lower bound on $|N(V')|$ in terms of $|V'|$ in the case of Steiner triple systems, while the second one seeks for edge-density conditions on triangle decompositions of general graphs $G = G(V, E)$, i.e. linear triple systems, where the condition $|N(V')| > 0$ must hold for all nontrivial subsets of V.

Problem 1.4 (Expander STSs)**.** Does there exist an infinite family of Steiner triple systems STS(n) such that for some $\varepsilon > 0$, $\frac{|N(V')|}{|V'|} \geq \varepsilon$ for every nontrivial $V' \subset V(G)$ provided that $|V'| \leq |V|/2$? How large $\varepsilon > 0$ can be?

This can be interpreted as the analogue of the expander property of graphs and the vertex isoperimetric number [**1**]. Similar generalised concepts for expanding triple systems were introduced very recently by Conlon and his coauthors [**4, 5**], see also the related paper [**9**]. Observe however that his definition is slightly different for a triple system to be expander.

Problem 1.5 (Sparse spreading linear triple systems)**.** What is the minimum size $\xi_{sp}(n)$ of a linear spreading triple system $\mathcal{F}$ on n vertices?

For these triple systems, the closure of any nontrivial subset with respect to the underlying graph of the triple system is the whole system.

Note that one might require only a weaker condition, namely that the closure of any nontrivial subset of the triple system $\mathcal{F}$ (i.e. consisting of at least two triples) should be the whole system. For this concept, we introduce the following notation.

Notation 1.6. A triple system $\mathcal{F}$ is *weakly spreading* if $\mathrm{cl}(V') = V$ for every

$$V' = V(\mathcal{F}') \ : \ \mathcal{F}' \subseteq \mathcal{F}, \ |\mathcal{F}'| > 1.$$

Problem 1.7 (Sparse weakly spreading linear triple systems)**.** What is the minimum size $\xi_{wsp}(n)$ of a linear weakly spreading triple system $\mathcal{F}$ on n vertices?

Our main results are as follows.

Theorem 1.8. *For odd prime number p, there exists a Steiner triple system* STS$(3p)$ *of order $3p$, for which*

$$|N(V')| \geq |V'| - 3$$

for every $V' \subset V(G)$ of size $|V'| \leq |V|/2$.

The result is clearly sharp.

Corollary 1.9. *For every sufficiently large n, there exists a Steiner triple system* STS$(\overline{n})$ *of order $\overline{n}$, for which*

$$|N(V')| \geq |V'| - 3$$

for every $V' \subset V(G)$ of size $|V'| \leq |V|/2$, where $\overline{n} \in [n - n^{0.525}, n]$.
Consequently, for every n one can find a Steiner triple system $\mathcal{S}$ of size $|\mathcal{S}| = (1 + o(1))\frac{n^2}{6}$ which is almost 1-expander.

As we will see, much smaller edge density compared to that of STSs' still enables us to construct spreading linear triple systems.

Theorem 1.10. *For the minimum size of a spreading linear triple system, we have*

$$0.1103n^2 < \xi_{sp}(n) < \Big(\frac{5}{36} + o(1)\Big)n^2 \approx 0.139n^2$$

Surprisingly, the weak spreading property does not require a dense structure at all.

Theorem 1.11. *For the minimum size of a weakly spreading linear triple system, we have*

$$n - 3 \le \xi_{wsp}(n) < \frac{8}{3}n + O(1)$$

The extended abstract is organized as follows. In Section 2 we present the proof ideas and the key constructions, while in Section 3 we discuss some related problems.

2. Proof sketches and constructions

Proof sketch of Theorem 1.10, lower bound. Let $\mathcal{H}$ be a spreading linear triple system. We use double counting on the number of 4-vertex subgraph F of the shadow $G = G(\mathcal{H})$ which is obtained from a triple and a vertex adjacent to exactly one vertex of the triple in G. We get

$$\sum_{v \in V(G)} \binom{\overline{d}(v)}{2} = \sum_{T \in \mathcal{H}} \mathrm{Val}(T) \tag{1}$$

where the *value of the triple* T, $\mathrm{Val}(T)$ denotes the number of F subgraphs corresponding to the triple T and $\overline{d}(v)$ denotes the degree of v in $\overline{G}$. In general, the bound $\mathrm{Val}(T) \le n - 3$ is sharp. However, it can be demonstrated by a rather involved argument on local-global structure and convex optimisation that the average value of the triples cannot exceed $0.5183n$, from which the theorem follows. Here $\tau \approx 0.5183$ is the unique local extremum of the rational function $\frac{z(1-z)(3-2z)}{4z^2-6z+3}$ in the interval $z \in [1/2, 1]$. $\square$

Proof of Theorem 1.11, lower bound. Take an arbitrary triple T_1 of the weakly spreading system $\mathcal{F}$. Observe that there must exist a triple T_2 sharing a common vertex with T_1, otherwise their union would violate the weakly spreading condition. From now on, the weakly spreading condition guarantees the existence of an ordering of the triples $T_1, T_2, \dots T_m$ of $\mathcal{F}$, such that

$$|T_k \cap \bigcup_{i=1}^{k-1} T_i| \ge 2 \quad (\forall k \le m).$$

This in turn implies the lower bound. Notice that it is sharp for $n = 5, 6, 7$. $\square$

The upper bound of Theorem 1.10 follows from the construction described in the Appendix. It applies the Cauchy–Davenport theorem and the key idea is to take a triangle decomposition of a well structured dense graph G which in our case is a perturbation of

$$K_{6k} \setminus \left(K_{k,k} \dot{\bigcup} K_{k,k} \dot{\bigcup} K_{k,k} \right).$$

The upper bound of Theorem 1.11 is derived from the upper bound of Theorem 1.10 by the following construction.

Construction 2.1. Consider a spreading linear triple system $\mathcal{H}$ on n vertices and $\xi_{sp}(n) = \frac{1}{3}\left(\binom{n}{2} - Cn^2\right)$ triples, with the appropriate constant C. Assign a new vertex $v(xy)$ to every not-covered edge xy of the underlying graph $G = G(\mathcal{H})$, and add newly formed triples by taking $\{x, y, v(xy)\} : xy \notin G\}$.

Proposition 2.2. *Construction 2.1 provides a weakly spreading system on* $n+Cn^2$ *vertices with* $\frac{1}{3}\left(\binom{n}{2} + 2Cn^2\right)$ *triples, hence we obtain*

$$\xi_{wsp}(N) \le \frac{2}{3}N + \frac{1}{6C}N.$$

Proof sketch of Theorem 1.8. Consider the Steiner triple system construction by Bose and Skolem on $6k+3$ vertices, where $2k+1$ is a prime number [**14**], where the ground set is partitioned into 3 equal parts $A \cup B \cup C$. The structure of the triples implies that one can bound the neighbourhood of $A_0 \cup B_0 \cup C_0 \subset A \cup B \cup C$ by applying the Cauchy–Davenport and the Erdős–Heilbronn theorems [**15**]. $\square$

3. Discussion

We mention first the obvious connection between the spreading and weakly spreading properties and connectivity of 3-graphs.

Definition 3.1. A 3-uniform hypergraph $\mathcal{F}$ is *strongly connected* if every vertex partition $U \dot\cup (V \setminus U)$ induces a triple T with $|T \cap U| = 2$, provided $|U| \ge 4$.

The latter definition implies that if the partition classes U and $(V \setminus U)$ are large enough, then triples of type $|T \cap U| = 2$ and $|T \cap U| = 1$ both should appear. The condition $|U| \ge 4$ enables us to apply this concept in linear 3-graphs. We note the spreading property is stronger than the strong connectivity, while the weakly spreading connectivity is weaker.

Observation 3.2. A Steiner triple system is subsystem-free, that is, spreading if and only if it is strongly connected. Every spreading linear triple system is strongly connected. Every strongly connected 3-graph is weakly spreading.

In operation research, related concepts are the connectivity of directed hypergraphs, and backward and forward (hyper)graphs, see e.g. [**10**].

A Latin square of order n is an $n \times n$ matrix in which each one of n symbols appears exactly once in every row and in every column. A subsquare of a Latin square is a submatrix of the Latin square which is itself a Latin square. Note that Latin squares of order n and 1-factorizations of complete bipartite graphs $K_{n,n}$ are corresponding objects.

Our results are strongly related to theorems on subsquare-free Latin squares as well. Indeed, these objects provide constructions in our case (after suitable transformation), while the expander property yields possible generalisations of such results. We mention here only

Theorem 3.3 ([**2, 13**, Maenhaut, Wanless, Webb]). *Subsquare-free Latin squares exists for every odd order.*

(Note that for prime order the statement follows from the Cauchy-Davenport theorem if we take the Cayley table of the additive group $\mathbb{Z}_p$.)

We also point out a geometric relation to these results. We are given a set of collinearity conditions for point triples and seek for conditions on possible embedding of the structure to a projective plane of given order. Here the spreading property for the set of triples would imply that the triples correspond either to distinct lines, or to one common line.

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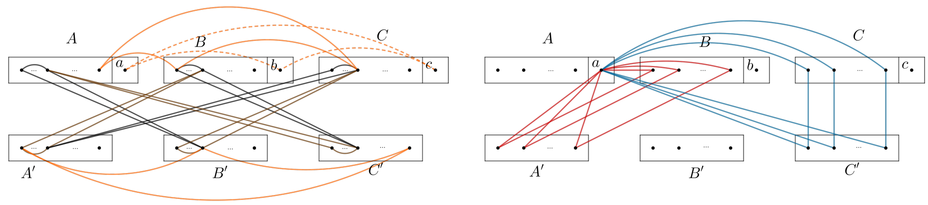
4. Appendix

We will construct a spreading triple system $\mathcal{H}$ on $n = 6p + 3$ vertices for every p such that p is an odd prime number, with $|E(G(\mathcal{H}))| \approx \frac{5}{12}n^2$.



(a) Black, brown and orange triples and $\{a, b, c\}$ **(b)** Red and blue triples through a

Figure 1. Overview of the triple types in Construction 4.1

Construction 4.1. The vertex set of $\mathcal{H}$ is the disjoint union of 6 smaller subsets, namely $V = A \cup B \cup C \cup A' \cup B' \cup C'$, where $|A| = |B| = |C| = p + 1$ and $|A'| = |B'| = |C'| = p$. Denote the elements of A with $a_0, a_1, \dots, a_{p-1}$ and a special vertex a. Similarly $B = \{b_0, b_1, \dots, b_{p-1}, b\}$ and $C = \{c_0, c_1, \dots, c_{p-1}, c\}$. For A', B', C' we note the corresponding vertices by α, β, γ respectively, and index their elements again from 0 up to $p - 1$. The set of triples in $\mathcal{H}$ are defined as follows:

- black triples:
  - between A and B': $\{a, a_j, \beta_j\}$ (for $0 \le j \le p-1$); and $\{a_i, a_{2j-i \pmod p}, \beta_j\}$ (for $0 \le i \ne j \le p-1$)
  - between B and C': $\{b, b_j, \gamma_j\}$ (for $0 \le j \le p-1$); and $\{b_i, b_{2j-i \pmod p}, \gamma_j\}$ (for $0 \le i \ne j \le p-1$)
  - between C and A': $\{c, c_j, \alpha_j\}$ (for $0 \le j \le p-1$); and $\{c_i, c_{2j-i \pmod p}, \alpha_j\}$ (for $0 \le i \ne j \le p-1$)
- brown triples:
  - between A' and B: $\{\alpha_i, \alpha_{2j-i \pmod p}, b_j\}$ (for $0 \le i \ne j \le p-1$)
  - between B' and C: $\{\beta_i, \beta_{2j-i \pmod p}, c_j\}$ (for $0 \le i \ne j \le p-1$)
  - between C' and A: $\{\gamma_i, \gamma_{2j-i \pmod p}, a_j\}$ (for $0 \le i \ne j \le p-1$)
- orange triples:
  - between $A \setminus \{a\}$, $B \setminus \{b\}$ and $C \setminus \{c\}$: $\{a_i, b_j, c_{i+j \pmod p}\}$ (for $0 \le i, j \le p-1$)
  - between A', B' and C': $\{\alpha_i, \beta_j, \gamma_{i+j+1 \pmod p}\}$ (for $0 \le i, j \le p-1$)
  - $\{a, b, c\}$
- red triples:
  $\{a, \alpha_j, b_j\}$, $\{b, \beta_j, c_j\}$ and $\{c, \gamma_j, a_j\}$ (for $0 \le j \le p-1$)
- blue triples:
  $\{a, \gamma_j, c_j\}$, $\{b, \alpha_j, a_j\}$ and $\{c, \beta_j, b_j\}$ (for $0 \le j \le p-1$)

Figure 1 describes the type of triples in $\mathcal{H}$.

Theorem 4.2. *The triple system $\mathcal{H}$ defined above has the spreading property.*

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