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k-HYPERGRAPHS WITH REGULAR AUTOMORPHISM GROUPS

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Abstract. Regular representations of finite groups, as introduced by Cayley, are among the most natural permutation representations of finite groups. Thus, the question which regular representations appear as full automorphism groups of combinatorial structures has been addressed and resolved for several classes of structures, notably for graphs (where they are called Graphical Regular Representations, GRR's), digraphs (Digraphical Regular Representations, DRR's) as well as for hypergraphs allowing for hyperedges of varying sizes. In the present paper, we focus on k-hypergraphs, which are hypergraphs in which all hyperedges are of the same size k, and address the question which k-regular hypergraphs possess full automorphism groups acting regularly on the vertices. We rely on the concept of a Cayley hypergraph (defined here) and show that all sufficiently large finite groups admit a regular representation as the full automorphism group of a 3-hypergraph.

1. Introduction

All the groups considered in our paper are finite, and so are the sets upon which they act. A transitive action of a group G on a set X is *regular* if the stabilizer of any vertex is trivial. The left regular representation of G acting on itself will be denoted $G_L = \{\sigma_g : G \to G \mid g \in G, \sigma_g(h) = gh, \text{ for all } h \in G\}$, and every regular action of G on a set X is equivalent to the action of G_L on G.

Frucht proved in his 1938 paper [**3**] that every finite group G is *isomorphic* to the full automorphism group of some finite graph. However, the actions of these groups on the vertices of the corresponding graphs are far from being regular (or even transitive). To address this, the *Graphical Regular Representation Problem* (the GRR problem) asks for the classification of finite groups G for which there exists a graph $\Gamma = (V, E)$ whose full automorphism group *acts regularly* on its vertex set V *and is isomorphic* to G. Equivalently, the GRR problem asks for the classification of finite groups G that admit an edge set $E \subseteq \mathcal{P}_2(G)$ with the property that the full automorphism group of the graph (G, E) equals G_L (we use the notation $\mathcal{P}_k(V)$ to denote the *set of all k-element subsets of* V). Such groups

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are said to *admit a GRR* and include almost all finite groups with the exception of abelian groups of exponent at least 3, generalized dicyclic groups, and thirteen sporadic groups of order not exceeding 32 [**12, 4, 5, 9, 10**]. In a variation of this problem, the *Digraphical Regular Representation Problem* (DRR) addresses the same question for digraphs with the classification showing that finite groups admitting DRR's include all finite groups but $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathbb{Z}_3^2$ and the quaternion group $\mathbb{Q}_8$ [**1**].

In our paper, we consider *k-uniform hypergraphs*, (also called k-hypergraphs), and ask which finite groups G admit a k, $0 \leq k \leq |G|$, such that there exists a set of k-hyperedges $\mathcal{H} \subseteq \mathcal{P}_k(G)$ with the property that the full automorphism group of the k-hypergraph $(G, \mathcal{H})$ equals G_L. The majority of our results concern the case $k = 3$; while the case $k = 2$ is clearly the original GRR problem. The cases $k = 3$ and $k = 4$, however in the much more specialized setting of Steiner triple and quadruple systems, have also been considered in [**8**].

The more general case of this question concerning hypergraphs with hyperedges of varying sizes has already been settled in [**6**], where it has been shown that a hypergraph (but not necessarily a k-uniform hypergraph) whose full automorphism group is equal to the left regular representation G_L of G exists for all finite groups but $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, and $\mathbb{Z}_2^2$. The results in [**6**] rely heavily on the hyperedges being of varying sizes.

2. Preliminaries

A *hypergraph* $\Gamma = (V, \mathcal{H})$ consists of a set V and a collection $\mathcal{H}$ of subsets of V. A hypergraph is said to be *k-uniform* if all the subsets in $\mathcal{H}$ are of the size k, i.e., $\mathcal{H} \subseteq \mathcal{P}_k(V)$. We will call the elements from $\mathcal{H}$ *hyperedges*, or more specifically, *k-hyperedges* of Γ. The *automorphism group* of a k-hypergraph $\Gamma = (V, \mathcal{H})$, denoted $\mathrm{Aut}(\Gamma)$, is the group of permutations of V that preserve the k-hyperedges, i.e., permutations $\varphi \in \mathrm{Sym}_V$ with the property $\varphi(H) \in \mathcal{H}$, for all $H \in \mathcal{H}$. A finite group G admits a *regular representation as the full automorphism group of a k-uniform hypergraph* if there exists a set of k-hyperedges $\mathcal{H} \subseteq \mathcal{P}_k(G)$ for which $\mathrm{Aut}(G, \mathcal{H}) = G_L$.

In what follows, we rely on the following lemma that generalizes Sabidussi's famous characterization of Cayley graphs [**11**]. Given a k-subset H of G, H^G denotes the family of k-subsets $\{\, gH \mid g \in G\}$. The proof of this lemma is a special case of the proof of a slightly more general result proved in [**6**].

Lemma 2.1 ([**6**]). *Let $\Gamma = (V, \mathcal{H})$ be a vertex-transitive k-uniform hypergraph. Then Γ admits a regular subgroup G of the full automorphism group $\mathrm{Aut}(\Gamma)$ if and only if there exists a family of k-sets $H_r \in \mathcal{P}_k(G)$, $1 \leq r \leq s$, all of which contain 1_G, such that Γ is isomorphic to $(G, \bigcup_{r=1}^{s} H_r^G)$.*

Lemma 2.1 makes the four special cases $k \in \{0, 1, |G| - 1, |G|\}$ easy to deal with. For example, if the full automorphism group of a 1-hypergraph $(G, \mathcal{H})$ were equal to G_L, any non-empty set of 1-hyperedges (i.e., hyperedges consisting of a single vertex) would have to contain $\{1_G\}^G = \{\{g\} \mid g \in G\}$, and would therefore consist of all 1-element subsets of G. Since the full automorphism groups of both $(G, \mathcal{P}_1(G))$ and $(G, \emptyset)$ are equal to Sym_G, and Sym_G acts regularly on G if and only if $|G| = 1$ or $|G| = 2$, the only groups that can be represented as regular automorphism groups of a 0- or 1-uniform hypergraph are the groups $\mathrm{Sym}_{\{1\}} \cong \mathbb{Z}_1$ and $\mathrm{Sym}_{\{1,2\}} \cong \mathbb{Z}_2$. Clearly, since $\mathrm{Aut}(V, \mathcal{H}) = \mathrm{Aut}(V, \{V \setminus H \mid H \in \mathcal{H}\})$ [**6**], the same is true for $k = |G| - 1$ and $k = |G|$.

The cases $k = 2$ or $k = |G| - 2$ are covered by the classification of GRR's, and a finite group G can be represented as a full automorphism group of a 2- or a $(|G| - 2)$-uniform hypergraph if and only of G does admit a GRR. Thus, from now on, we shall assume $3 \leq k \leq |G| - 3$.

The case of cyclic groups $\mathbb{Z}_n$ is a typical example of the kind of results we are looking for. The groups $\mathbb{Z}_1$ and $\mathbb{Z}_2$ have trivially a regular representation via a k-regular hypergraph (for all admissible k's). On the opposite side, none of the groups $\mathbb{Z}_n$, $n \geq 3$, can be regularly represented as the full automorphism of a k-hypergraph with $k \in \{0, 1, 2, n-2, n-1, n\}$, by the discussion following Lemma 2.1 and the characterization of groups admitting GRR's [**4, 12**]. Moreover, the groups $\mathbb{Z}_n$, $n = 3, 4, 5$, cannot be regularly represented on any hypergraph at all [**6**]. All the remaining cases are covered by the following theorem (the proof uses hyperedges of two types: $\{j, j+1, \ldots, j+k-1\} \mid 1 \leq j \leq n\}$ and $\{\, j, j+1, \ldots, j+k-2, j+k\} \mid 1 \leq j \leq n\})$.

Theorem 2.2. *A cyclic group $\mathbb{Z}_n$, $n \geq 6$, admits a regular representation on a k-uniform hypergraph $(\mathbb{Z}_n, \mathcal{H})$ if and only if $3 \leq k \leq n-3$.*

3. Cayley hypergraphs

In order to classify finite groups G that allow for the existence of a k-hypergraph $(G, \mathcal{H})$, $k \geq 3$, satisfying $\mathrm{Aut}(G, \mathcal{H}) = G_L$, we introduce the following generalization of Cayley graphs.

Let G be a (finite) group, and let $X_1, X_2, \ldots, X_{k-1}$ be subsets of G that do not contain the identity 1_G. The *Cayley k-hypergraph*

$$C_k(G; X_1, X_2, \ldots, X_{k-1})$$

is the incidence structure $(G, \mathcal{H})$ with $\mathcal{H}$ being the set of all k-subsets of the form

$$\{g, gx_1, gx_1x_2, \ldots, gx_1x_2 \ldots x_{k-1}\},$$

$g \in G$, and $x_i \in X_i$, for $1 \leq i \leq k-1$. Note that we strictly require that the hyperedges have exactly k vertices in order to be included, i.e., all the group elements $g, gx_1, gx_1x_2, \ldots, gx_1x_2 \ldots x_{k-1}$ must be different. This is equivalent to saying $x_i x_{i+1} \ldots x_j \neq 1$, for all $1 \leq i \leq j \leq k-1$. (This requirement may occasionally force $\mathcal{H} = \emptyset$.) The 2-hypergraph $C_2(G; X)$ is the Cayley graph $C(G, X)$, and in the case when $X = X_1 = X_2 = \cdots = X_{k-1}$, the resulting hyperedges of $C_k(G; X, X, \ldots, X)$ are the sets of vertices corresponding to the *k-arcs* of the Cayley graph $C(G, X)$ ([**2**, Chapter 17]) that contain no repeated vertices.

4. k-UNIFORM REGULAR REPRESENTATIONS OF NON-CYCLIC GROUPS, $k \geq 3$

The automorphism group of a k-hypergraph $C_k(G; X_1, X_2, \ldots, X_{k-1})$ should obviously be related to the groups $\mathrm{Aut}(C(G, X_i))$. For instance, since graph automorphisms preserve k-arcs,

$$\mathrm{Aut}(C(G, X)) \leq \mathrm{Aut}(C_k(G; X, X, \ldots, X)).$$

The next lemma presents sufficient conditions for this inclusion to be an identity. The *girth* of a graph $\Gamma = (V, \mathcal{E})$ is the number of edges in a smallest cycle in Γ.

Lemma 4.1. *Let $k \geq 2$ be an integer, and $C(G, X)$ be a Cayley graph of girth $g > 2k - 2$ and valency $|X| > k - 1$. Then*

$$\mathrm{Aut}(C(G, X)) = \mathrm{Aut}(C_k(G; X, X, \ldots, X)).$$

The proof of this lemma uses the fact that graphs of large girth are locally isomorphic to trees.

Corollary 4.2. *If a finite group G admits a GRR of girth $g > 2m - 2$ and valency r, then G can be regularly represented as the full automorphism group of some k-hypergraph for all $2 \leq k \leq \min\{m, r - 1\}$.*

In particular, any finite group G that admits a GRR of valency at least 4 and not containing 3- or 4-cycles, admits a 3-uniform regular representation.

The following technical lemma presents a way of avoiding the need for high girth. A set $\emptyset \neq X \subseteq G$ is said to be *symmetric* if it is closed under inverses, i.e., $X = X^{-1} = \{x^{-1} \mid x \in X\}$. A *reduced product* $x_1 x_2 \ldots x_\ell$ is one that does not contain a factor followed by its inverse; $x_i \neq x_{i+1}^{-1}$, for $1 \leq i < \ell$.

Lemma 4.3. *Let G be a finite group, $X_1, X_2, \ldots X_{k-1}$ be symmetric subsets of G not containing 1_G, $|X_i| \geq k$, for all $1 \leq i \leq k - 1$, and suppose that all reduced products $x_1 x_2 \ldots x_\ell$, $x_i \in X_i$, $1 \leq \ell \leq k - 1$, represent different elements of G. Then, $\mathrm{Aut}(C_k(G; X_1, X_2, \ldots, X_{k-1})) \leq \mathrm{Aut}(C(G, X_1))$.*

In order to use Corollary 4.2, it is generally to one's advantage when the valency of the GRR is large. On the other hand, the next lemma gives the best results when the valency is small.

Lemma 4.4. *Let G be a finite group that is not cyclic and admits a GRR $C(G, X)$. Then G admits a regular representation as the full automorphism group of a k-hypergraph for all $k \geq 2$ satisfying the inequality $\sum_{j=1}^{k-1} (|X| + 1)^{2j} \leq |G|$.*

In particular, any finite group G of order greater than or equal to $4^2 + 4^4 = 272$ that admits a GRR of valency 3 admits a 3-uniform regular representation.

The proof of Lemma 4.4 relies on Lemma 4.3 and a recursive construction of the generator sets $X_1, X_2, \ldots$.

The two infinite classes of finite groups that do not admit a GRR are the finite abelian groups of exponent at least 3 and generalized dicyclic groups. Most of them allow for a regular representation via a 3-hypergraph.

Lemma 4.5. *Let G be a finite abelian group that contains a cyclic subgroup of order at least* 6 *or let G be a finite generalized dicyclic group with a normal abelian subgroup A of index* 2 *that contains a cyclic subgroup of order at least 6. Then G can be regularly represented as the full automorphism group of some* 3*-hypergraph on G.*

To prove the main result of our paper, we need one more lemma. A generating set X of a group G is said to be *irreducible* if no element from X can be omitted while X remains a generating set for G.

Lemma 4.6. *Let G be a finite group that admits an irreducible generating set X of size at least* 4*. Then G admits a regular representation as the full automorphism group of some* 3*-hypergraph.*

Combining the lemmas presented in our paper with the classification of finite groups contained in [**12, 4**] yields the main theorem of our paper.

Theorem 4.7. *A finite group G that admits neither a GRR nor a regular representation via a* 3*-hypergraph must be one of the following groups*

$$\mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_5 \text{ and } \mathbb{Z}_2^2, \tag{1}$$

$$\mathbb{D}_3,\ \mathbb{Q}_8 \text{ and } \mathbb{Z}_2^3, \tag{2}$$

$$\mathbb{Z}_4^3,\ \mathbb{Z}_5^3 \text{ and } \mathbb{D}_5 \times \mathbb{Z}_5. \tag{3}$$

We already know that the groups from the list (1) do not admit a regular representation via any hypergraph [**6**]. An exhaustive search run by Martin Mačaj [**7**] determined that all finite groups of orders greater than or equal to 6 and smaller than or equal to 32 but the groups on the list (2) admit a regular representation via a 3-hypergraph. The groups on (2) admit neither a GRR nor a regular representation via a 3-hypergraph [**12, 4, 7**]. We are currently working on a proof that all finite groups of order greater than 32 admit a regular representation via a 2- or a 3-hypergraph. Thus, we believe that we will be able to prove that the only finite groups that do not admit a regular representation via a 2- or 3-hypergraph are those on the lists (1) and (2).

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