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t-STRONG CLIQUES AND THE DEGREE-DIAMETER PROBLEM

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Abstract. For a graph G, $L(G)^t$ is the t-th power of the line graph of G – that is, vertices of $L(G)^t$ are edges of G and two edges $e, f \in E(G)$ are adjacent in $L(G)^t$ if G contains a path with at most t vertices that starts in a vertex of e and ends in a vertex of f. The *t-strong chromatic index* of G is the chromatic number of $L(G)^t$ and a *t-strong clique* in G is a clique in $L(G)^t$. Finding upper bounds for the *t-strong chromatic index* and *t-strong clique* are problems related to two famous problems: the conjecture of Erdős and Nešetřil concerning the strong chromatic index and the degree/diameter problem.

We prove that the size of a t-strong clique in a graph with maximum degree Δ is at most $1.75\Delta^t + O\left(\Delta^{t-1}\right)$, and for bipartite graphs the upper bound is at most $\Delta^t + O\left(\Delta^{t-1}\right)$. We also show results for some special classes of graphs: $K_{1,r}$-free graphs and graphs with a large girth.

1. Introduction

Let G be a graph. By $L(G)^t$ we denote the t-th power of the line graph of G – that is, vertices of $L(G)^t$ are edges of G and two edges $e, f \in E(G)$ are adjacent in $L(G)^t$ if G contains a path with at most t vertices that starts in a vertex of e and ends in a vertex of f. The *t-strong chromatic index* of G is the chromatic number of $L(G)^t$ and a *t-strong clique* in G is a clique in $L(G)^t$.

The underlying motivation of our research is to determine the extremal value of t-strong chromatic index of a graph with given maximum degree, in the setting where t is treated as a constant and maximum degree is allowed to be arbitrarily large.

Problem 1. Find the smallest constant b_t such that for every graph G of sufficiently large maximum degree Δ, the t-strong chromatic index of G is at most $b_t\Delta^t$.

Problem 1 have been extensively studied for $t = 2$.[1] A greedy argument shows that $b_t \leq 2$; Erdős and Nešetřil asked in 1985 whether the bound on b_2 can

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[1] 2-strong chromatic index is commonly known as strong chromatic index; we refrain from using this term for consistency.

be replaced by $2 - \varepsilon$ for any positive ε [**7**]. Molloy and Reed gave a positive answer in 1997 [**14**]; recently, it was improved by Bruhn and Joos [**3**] and Bonamy, Perrett and Postle [**2**] and the current record in $b_2 \leq 1.835$. Since the best known construction shows that $b_2 \geq 1.25$, the problem remains wide open. See [**17**] for some more specific results and problems regarding bounds on the 2-strong chromatic index.

Much less is known for $t > 2$. Kaiser and Kang proved in 2014 that $b_t \leq 2 - \varepsilon$ for $\varepsilon \approx 0.00008$ [**11**], and it remains the best known. On the other hand, there are specific constructions of bipartite graphs witnessing that $b_3, b_4, b_6 \geq 1$, but the general lower bound on b_t goes to 0 with t goes to infinity.

In this paper we focus on a relaxation of Problem 1, where the t-strong chromatic index is replaced by the size of a maximum t-strong clique.

Problem 2. Find the smallest constant c_t such that for every graph G of sufficiently large maximum degree Δ, the maximum size of a t-strong clique in G is at most $c_t\Delta^t$.

It is not clear how close to each other are solutions of Problems 1 and 2. The size of a maximum t-strong clique can be much smaller than the t-strong chromatic index for some graphs, but it may not be the case with their extremal values. In fact, the best known lower bounds for b_t come from constructions of t-strong cliques, i.e. the same lower bound holds for c_t.

Problem 2 have already been studied for $t = 2$ [**8, 18**]; we know that $c_2 \leq \frac{4}{3}$ (recall that $b_2 \leq 1.835$). We are not aware of any nontrivial upper bounds for $t > 2$.

Our topic has somewhat similar flavor to the degree/diameter problem, that asks for the maximum possible number $n_{\Delta,D}$ of vertices in a graph of maximum degree Δ and diameter D (i.e. we restrict our attention to line graphs, but require the t-th power only to contain a clique, instead of being a clique). Clearly $n_{\Delta,D} \leq (1 + o(1))\Delta^D$, but it is immensely difficult to either improve this upper bound by more than an additive constant or give a lower bound tight up to a multiplicative constant; see a survey by Miller and Širáň [**13**].

This connection provides us with an improved lower bound on c_t and b_t. The authors in [**11**] refer to Proposition 1 from [**12**] as the best known lower bound; it says that $b_t \geq \frac{1}{2(t-1)^{t-1}}$. However, we have $c_t \geq \frac{1}{2} \limsup_{\Delta\to\infty} \frac{n_{\Delta,t-1}}{\Delta^{t-1}}$ (to see this, take a graph with maximum degree Δ, diameter $t-1$ and $n_{\Delta,t-1}$ vertices, add edges so that almost every vertex has degree Δ, and note that edge set of the resulting graph forms a t-strong clique of order $\frac{1}{2}\Delta n_{\Delta,t-1}$). Together with the result of Canale and Gómez, that $n_{\Delta,D} \geq \left(\frac{\Delta}{(1.59)}\right)^D$ for sufficiently large D and infinitely many values of Δ [**4**], it implies that for sufficiently large t, $c_t \geq \frac{1}{2}\left(\frac{1}{1.59}\right)^{t-1}$.

Our main contribution is an upper bound of 1.75 on c_t.

Theorem 3. *Let G be a graph with maximum degree Δ. For every $t \geq 2$, the size of a t-strong clique in G is at most $1.75\Delta^t + O\left(\Delta^{t-1}\right)$.*

We also improve the constant to 1 for bipartite graphs. Recall that for $t = 3, 4, 6$ it matches known lower bounds, that are attained from constructions of bipartite graphs. It is also generalization of the earlier result for $t = 2$ [**9**].

Theorem 4. *Let G be a bipartite graph with maximum degree Δ. For every $t \geq 2$, the size of a t-strong clique in G is at most $\Delta^t + O\left(\Delta^{t-1}\right)$.*

Theorems 3 and 4 bound the size of a t-strong clique by an absolute constant times Δ^t. For claw-free graphs, and $K_{1,r}$-free graphs in general, we can prove a bound with the constant that goes to 0 with t goes to infinity. An analogous result holds for the degree/diameter problem [**6**].

Proposition 5. *Let G be $K_{1,r}$-free graph with maximum degree Δ. For every $t \geq 2$, the size of a t-strong clique in G is at most $2\left(\frac{r-2}{r-1}\right)^{t-2}\Delta^t + O\left(\Delta^{t-1}\right)$.*

The t-strong chromatic index drops down to $O\left(\frac{\Delta^t}{\log \Delta}\right)$ in graphs of girth at least $2t + 1$. This is tight up to a multiplicative constant (dependent on t), because there are graph of arbitrarily large girth and t-strong chromatic index at least $\Theta\left(\frac{\Delta^t}{t \log \Delta}\right)$; see [**11**, Theorem 1.2 and Proposition 1.3]. We prove that for t-strong cliques, this drop is more steep: graph of girth $2t + 1$ have t-strong cliques of size at most $O\left(\Delta^{t-1}\right)$ (it is also a consequence of [**11**, Lemma 3.1]) and whenever the bound on girth is increased by 2, the order of magnitude decreases by Δ.

Theorem 6. *Let G be a graph with maximum degree Δ and girth at least $2t + 2x + 1$, for $t \geq 2$ and $0 \leq x \leq \left\lfloor \frac{t}{2} \right\rfloor - 1$. The size of a t-strong clique in G is at most $2^{t+2}\Delta^{t-x-1}$.*

2. Final remarks

Theorem 3 is probably not tight. In fact, we do not know if it is even tight up a multiplicative constant – known lower bound on c_t goes to 0 with t goes to infinity. However, recall that $c_t \geq \frac{1}{2} \limsup\limits_{\Delta \to \infty} \frac{n_{\Delta,t-1}}{\Delta^{t-1}}$; it implies that a proof that c_t is strictly smaller than $\frac{1}{2}$ for large t would yield a huge breakthrough in the degree/diameter problem.

A similar remark applies for claw-free graphs – Proposition 5 is certainly not tight, but improving it by a factor of more than 16 for claw-free graphs would imply the same breakthrough for general graphs; it is demonstrated by the following proposition.

Proposition 7. *Let c be a constant such that for every sufficiently large Δ and t, a t-strong clique in a claw-free graph of maximum degree Δ has less than $c\frac{1}{2^t}\Delta^t$ edges. Then, for every sufficiently large Δ and t, $n_{\Delta,t} < 2c\Delta^t + 3c\Delta^{t-1} + \Delta$.*

Proof. It suffices show that if there exists a graph G with maximum degree Δ, diameter t and at least $2c\Delta^t + 3c\Delta^{t-1} + \Delta$ vertices, then there exists a triangle free graph H with maximum degree Δ' that contains a strong clique of size $c\frac{1}{2^{t'}}\Delta'^{t'}$, for $\Delta' = 2\Delta$ and $t' = t + 2$.

Let G be such graph. Let G' be a maximal supergraph of G with maximum degree Δ and the same set of vertices. We claim that G' has at least $2c\Delta^t + 3c\Delta^{t-1}$ vertices of degree Δ. Indeed, by maximality, every two vertices of degree less than Δ are adjacent in G' and, since their degree is at most $\Delta - 1$, there are at most Δ of them.

Let H be the line graph of G'. As such, H is claw-free and has maximum degree at most $\Delta' = 2\Delta$. Moreover, the diameter of H is at most $t+1$; it follows that all edges of H form a $(t+2)$-strong clique. Note that each vertex of degree Δ in G' corresponds to $\binom{\Delta}{2}$ edges of H. Therefore, the number of edges of H is at least $\left(2c\Delta^t + 3c\Delta^{t-1}\right)\binom{\Delta}{2}$. For $\Delta > 2$ it is at least $c\Delta^{t+2}$, which equals to $c\frac{1}{2^{t'}}\Delta'^{t'}$, as desired. □

We would like to know, how close to optimal is Theorem 6. The constant 2^{t+2} is clearly overestimated, but we suspect that the order of magnitude may be correct – it is clearly correct for $x = \lfloor \frac{t}{2} \rfloor - 1$ (as demonstrated by a regular tree of diameter $t+1$, with either one or two centres, depending on the parity of t). For smaller x we do not know, as we are generally unable to construct graphs with large girth and large t-strong clique. We feel that a construction matching the order of magnitude from Theorem 6 would be very informative, even for $x = \lfloor \frac{t}{2} \rfloor - 2$, and leave it as an open problem.

Theorem 3 would imply that $b_t \leq 1.875$ if Reed's conjecture was true [**16**], and gives a fractional result with this constant by the fractional version of Reed's conjecture, shown to be true [**15**, Theorem 21.7]. However, considering the apparent hardness of Reed's conjecture, we think that a more promising way of obtaining some progress in Problem 1 would be to rely on bounding the number of edges in a vertex neighborhood in $L(G)^t$; this strategy yielded aforementioned results for $t = 2$.

Throughout this paper we always assumed that the maximum degree Δ is large enough, disregarding small cases. However, the problem of bounding the 2-strong chromatic index is remains interesting for smaller Δ (see [**1, 5, 10**]), and so would be bounding t-strong chromatic index and t-strong cliques.

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